GR Primer for Graduate Students

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Outline

General Relativity

GR Summary - 2 statements:

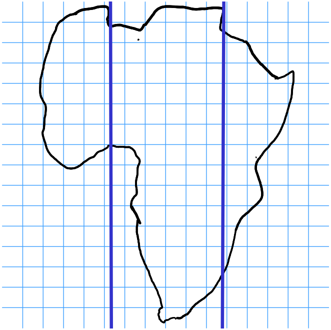
- Space-time is a curved pseudo-Riemannian manifold
- Matter curves space-time according to Einstein's equation:

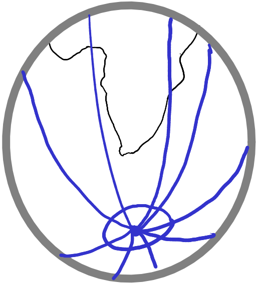
$$\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}}$$

- We need to a bit of mathematics to appreciate the summary
- Work in units where $c = 1$

Riemannian Geometry

- Euclid's 5^{th} postulate (E5)
  - roughly: parallel lines never meet
- Does not apply to spaces with curvature.





Lines of longitude meet at the South Pole

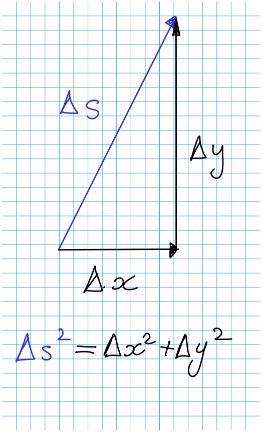
Riemannian Geometry

- Euclid's 5^{th} postulate (E5)
  - parallel lines never meet

  does not apply to spaces with curvature.
- The failure of E5 can be encoded in the metric
- The metric, ds^2, tells us the infinitesimal distance between points.
- Flat space metric:

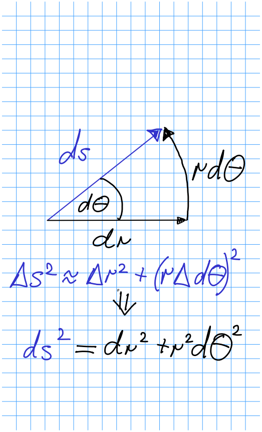
$$ds^2 = dx^2 + dy^2 + dz^2$$

is equivalent to E5



Metric

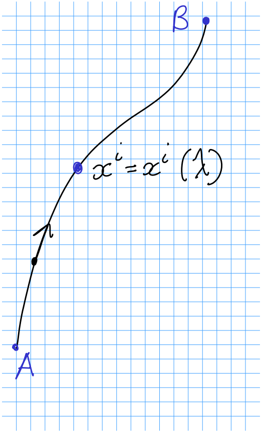
- Using the Einstein summation convention
  $ds^2 = g_{ij}dx^i dx^j \qquad i,j \in 1,2,3$
- In spherical coordinates
  $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$
  - $g_{ij} = \mathrm{diag}(1, r^2, r^2\sin^2\theta)$
- In a general curved space the metric is a function of position, $g_{ij} = g_{ij}(\vec{x})$
  - Eg for the sphere, $ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2)$



Metric and Distance

- Metric = infinitesimal distance
- Consider a curve $x^i(\lambda)$
- Integrate to get distance between points

$$s_{AB} = \int_A^B ds = \int_A^B d\lambda \frac{ds}{d\lambda}$$
$$= \int_A^B d\lambda \sqrt{g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}$$



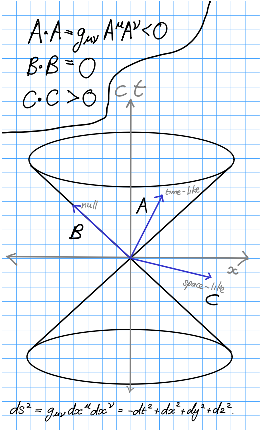
Space-time

- speed of light constant $\rightarrow$ special relativity

$\rightarrow$ space-time interval same for all inertial observers:

  - $d\tau^2 = -dt^2 + dx^2 + dy^2 + dz^2 = g_{\mu\nu} dx^\nu dx^\mu \quad \mu, \nu \in 0, 1, 2, 3$

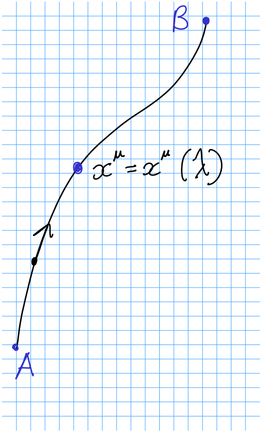
- The metric, $g_{\mu\nu}$, encodes the geometry of 4-d space-time
  - Flat space-time: $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$
- General curved pseudo-Riemannian geometry: $g_{\mu\nu} = g_{\mu\nu}(x^\alpha)$



Metric and Proper-time

- Metric in space-time also tells us how fast clocks run in space-time
- Consider a time-like curve $x^\mu(\lambda)$
- Integrate to get proper time measured by an observer

$$\tau_{AB} = \int_A^B \sqrt{-ds^2} = \int_A^B d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$



Metric Tensor

- Changing coordinates $x^\mu \to x^{\bar{\alpha}}$, distance between points must remain the same

$\Rightarrow$ $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \left(\frac{\partial x^\mu}{\partial x^{\bar{\alpha}}} dx^{\bar{\alpha}}\right) \left(\frac{\partial x^\nu}{\partial x^{\bar{\beta}}} dx^{\bar{\beta}}\right) = g_{\bar{\alpha}\bar{\beta}} dx^{\bar{\alpha}} dx^{\bar{\beta}}$

$\Rightarrow$

$$\boxed{g_{\bar{\alpha}\bar{\beta}} = \frac{\partial x^\mu}{\partial x^{\bar{\alpha}}} \frac{\partial x^\nu}{\partial x^{\bar{\beta}}} g_{\mu\nu}}$$

- Coordinate transformation property of metric
- E.g. in flat space, Cartesian $\to$ Spherical coordinates,

$$\mathrm{diag}(1,1,1) \to \mathrm{diag}(1, r^2, r^2 \sin^2\theta)$$

General Tensors

General Tensors transform according to the rule

$$S^{\bar{\alpha}\ldots}{}_{\bar{\beta}\bar{\gamma}\ldots}(\bar{x}) = \left(\frac{\partial x^{\bar{\alpha}}}{\partial x^{\mu}}\right)\cdots\left(\frac{\partial x^{\nu}}{\partial x^{\bar{\beta}}}\right)\left(\frac{\partial x^{\rho}}{\partial x^{\bar{\gamma}}}\right)\cdots S^{\mu\ldots}{}_{\nu\rho\ldots}(x)$$

- To remember the rule:
  - “Conservation of indices ” : number of free (unsummed) indices must balance
  - dummy (summed) indices in pairs.
  - Definition: An $\binom{n}{m}$ tensor has n upper and m lower indices.
  - A scalar is a $\binom{0}{0}$ tensor.

Why Tensors?

- Philosophy: Physics should not depend on the coordinate system we use.
- Tensor equations look the same in all coordinates:

$$S^{\bar{\alpha}\dots}{}_{\bar{\beta}\bar{\gamma}\dots} = 0 \Rightarrow S^{\mu\dots}{}_{\nu\lambda} = 0$$

$\Rightarrow$ Tensors are the natural language of classical physics.

Tensor gymnastics

Changing tensor type using the metric - Lowering Index

$$A^{\alpha\ldots}{}_{\mu\beta\ldots} = g_{\mu\nu} A^{\alpha\ldots\nu}{}_{\beta\ldots}$$

$\binom{n}{m}$-tensor (vector) $\rightarrow$ $\binom{n-1}{m+1}$-tensor

Definition: Inverse metric $g^{\mu\nu}$:

Matrix inverse of the metric $g^{\mu\nu} g_{\nu\alpha} = \delta^{\mu}_{\alpha}$

Changing tensor type using the Inverse metric- Raising Index

$$A_{\alpha\ldots}{}^{\mu\beta\ldots} = g^{\mu\nu} A_{\alpha\ldots\nu}{}^{\beta\ldots}$$

$\binom{n}{m}$-tensor (vector) $\rightarrow$ $\binom{n+1}{m-1}$-tensor

Covariant derivative

Notation

$$\frac{\partial}{\partial x^\mu} A = \partial_\mu A = A_{,\mu}$$

- Partial derivatives of scalar fields are $\binom{0}{1}$-tensors

$$\phi_{,\mu} = \left(\frac{\partial x^{\bar{\alpha}}}{\partial x^\mu} \right) \phi_{,\bar{\alpha}}$$

- Partial derivatives of other tensor fields are not tensors

$$A^\mu{}_{,\nu} = \frac{\partial x^{\bar{\alpha}}}{\partial x^\nu} \frac{\partial}{\partial x^{\bar{\alpha}}} \left(\frac{\partial x^\mu}{\partial x^{\bar{\beta}}} A^{\bar{\beta}} \right) = \frac{\partial x^{\bar{\alpha}}}{\partial x^\nu} \frac{\partial x^\mu}{\partial x^{\bar{\beta}}} A^{\bar{\beta}}{}_{,\bar{\alpha}} + \frac{\partial x^{\bar{\alpha}}}{\partial x^\nu} \frac{\partial^2 x^\mu}{\partial x^{\bar{\beta}} x^{\bar{\alpha}}} A^{\bar{\beta}}$$

Covariant derivatives

- Partial derivatives do not transform as tensors

$$A^{\mu}{}_{,\nu} = \frac{\partial x^{\bar{\alpha}}}{\partial x^{\nu}} \frac{\partial}{\partial x^{\bar{\alpha}}} \left(\frac{\partial x^{\mu}}{\partial x^{\bar{\beta}}} A^{\bar{\beta}} \right) = \frac{\partial x^{\bar{\alpha}}}{\partial x^{\nu}} \frac{\partial x^{\mu}}{\partial x^{\bar{\beta}}} A^{\bar{\beta}}{}_{,\bar{\alpha}} + \frac{\partial x^{\bar{\alpha}}}{\partial x^{\nu}} \frac{\partial^2 x^{\mu}}{\partial x^{\bar{\beta}} x^{\bar{\alpha}}} A^{\bar{\beta}}$$

  - Unwanted term proportional to to $A^{\bar{\beta}}$

→ Introduce a correction factor to cancel

  → Covariant derivative, ∇ which transforms as a tensor:

$$\nabla_{\mu} A^{\nu} = \partial_{\mu} A^{\nu} + \Gamma^{\nu}_{\mu\alpha} A^{\alpha}$$

  - This correction term, $\Gamma^{\mu}_{\alpha\beta}$ is called the connection

Covariant Derivative

- For the covariant derivative

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\alpha} A^\alpha$$

to be a $\binom{1}{1}$-tensor you can check the connection must transform as

$$\Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}} = \frac{\partial x^{\bar{\alpha}}}{\partial x^\rho} \frac{\partial x^\sigma}{\partial x^{\bar{\mu}}} \frac{\partial x^\lambda}{\partial x^{\bar{\nu}}} \Gamma^\rho_{\sigma\lambda} - \frac{\partial x^\rho}{\partial x^{\bar{\mu}}} \frac{\partial x^\sigma}{\partial x^{\bar{\nu}}} \frac{\partial^2 x^{\bar{\alpha}}}{\partial x^\rho x^\sigma}$$

- Not a tensor !

$\rightarrow$ Ugly 2nd terms cancels ugly term from partial derivative

Covariant derivative

- For a, $\binom{0}{1}$ tensor , ω_α, we can also make a covariant derivative

$$\nabla_\nu \omega_\mu = \partial_\nu \omega_\mu - \Gamma^\alpha_{\nu\mu}\omega_\alpha$$

- For a general $\binom{n}{m}$ tensor the rule is

$$\begin{aligned}\nabla_\alpha T^{\mu_1\mu_2\ldots\mu_n}{}_{\nu_1\ldots\nu_m} &= \partial_\alpha T^{\mu_1\mu_2\ldots\mu_n}{}_{\nu_1\ldots\nu_m} + \Gamma^{\mu_1}_{\alpha\lambda} T^{\lambda\mu_2\ldots\mu_n}{}_{\nu_1\ldots\nu_m} \\ &+ \Gamma^{\mu_2}_{\alpha\lambda} T^{\mu_1\lambda\ldots\mu_n}{}_{\nu_1\ldots\nu_m} + \ldots \\ &- \Gamma^{\lambda}_{\alpha\nu_1} T^{\mu_1\mu_2\ldots\mu_n}{}_{\lambda\ldots\nu_m} \ldots\end{aligned}$$

- Rule: $+\Gamma$ for each upper index and $-\Gamma$ for each lower index.

Christoffel Symbols

- What are these mysterious Γ's?
- Many choices
- Special choice = Christoffel Symbols

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\mu}(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu})$$

- Can check it transforms in the right way
- Also satisfies: $\nabla_\alpha g_{\mu\nu} = 0 \quad \nabla_\alpha g^{\mu\nu} = 0$
- Notice: In flat Cartesian coordinates $\Gamma^{\alpha}_{\mu\nu} = 0$ so

$$\nabla_\mu = \partial_\mu$$

  in that case.
- Will give some justification for Christofells later

Riemann Curvature Tensor

- Ordinary derivatives commute : $[\partial_\alpha, \partial_\beta] = 0$
- Failure of ∇'s to commute related to failure of E5
- Commutator of ∇'s encodes curvature.
- Riemann Curvature tensor

$$[\nabla_\alpha, \nabla_\beta] X^\gamma = R^\gamma{}_{\lambda\alpha\beta} X^\lambda$$

$\Rightarrow$

$$R^\gamma{}_{\lambda\alpha\beta} = \partial_\alpha \Gamma^\gamma_{\beta\lambda} - \partial_\beta \Gamma^\gamma_{\alpha\lambda} + \Gamma^\gamma_{\alpha\rho}\Gamma^\rho_{\beta\lambda} - \Gamma^\gamma_{\beta\rho}\Gamma^\rho_{\alpha\lambda}$$

Notation: Ricci Tensor

- The trace of the Riemann Tensor gives us Ricci Tensor

$$R_{\alpha\beta} = \sum_{\gamma=0...3} R^{\gamma}{}_{\alpha\gamma\beta}$$

$$R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta}$$

- The trace of the Ricci Tensor gives us the Ricci Scalar

$$R = R^{\lambda}{}_{\lambda} = g^{\lambda\nu} R_{\nu\lambda}$$

- The Riemann tensor has many symmetries and satisfies many identities (which follow from the definition)
- NB is the Bianchi identity

$$\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\alpha R_{\beta\lambda\mu\nu} + \nabla_\beta R_{\lambda\alpha\mu\nu} = 0$$

- This implies that the Einstein Tensor

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$$

satisfies

  - $\nabla_\mu G^{\mu\nu} = 0$

"Straight lines" in curved space

- How can we generalise the concept of a straight line to curved space?
  - Straight line = shortest distance two points

$\rightarrow$ Extremise the distance (or proper time)

$= \int |ds| = \int d\lambda \sqrt{\left| g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right|}$

$\Rightarrow$ Geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$$

- $\Gamma^\alpha_{\rho\sigma}$ = Christoffel symbols
- λ must be linearly related proper-distance/proper-time.

“Straight lines”

- In flat cartesian coordinates, $\Gamma^{\alpha}_{\beta\gamma} = 0$

$\rightarrow$ $\frac{d^2 x^{\mu}}{d\lambda^2} = 0$

$\rightarrow$ Straight-lines

- Newton’s equation

$$m\ddot{x}^i = F^i$$

Newton’s equation for gravity

$$m\ddot{x}^i = -m\partial^i(\phi_G) \Rightarrow \ddot{x}^i = -\partial^i(\phi_G)$$

- Geodesic equation

$$\ddot{x}^{\mu} = -\Gamma^{\mu}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}$$

- Christoffell symbols $\leftrightarrow$ Gravitational potential
- Works because inertial mass = gravitational mass

Gravity is not a force

- One of Einstein's great ideas is that gravity is not a force.
- No force $\rightarrow$ particles move on straight lines
- In curved space, straight lines $\rightarrow$ geodesics
- Gravity = curvature of space-time.
- No force + curved space-time $\rightarrow$ particles move geodesics
- Hard to reconcile with QFT. Wrong?

Mass-Energy curves space-time

- In Newton's theory mass is source of gravitational force
- Einstein: Gravity = curvature

$\rightarrow$ Mass-Energy curves space-time

- How? Need tensor description of matter

Stress-Energy Tensor

Stress-Energy Tensor definition

$T^{\mu\nu} =$ Flux of μ^{th} component of momentum in ν^{th} direction

- For example, in the rest frame of a perfect fluid in flat space-time,

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

- Conservation of mass-energy $\Leftrightarrow \nabla_\mu T^{\mu\nu} = 0$

GR

- Mass energy curves space-time
- $\nabla_\mu G^{\mu\nu} = 0$
- $\nabla_\mu T^{\mu\nu} = 0$
- Einstein guessed $G^{\mu\nu} \propto T^{\mu\nu}$
- To recover Newtonian gravity need,

$$G^{\mu\nu} = 8\pi G T^{\mu\nu}$$

- $\nabla_\mu g^{\mu\nu} = 0$, so can also add term $\propto g^{\mu\nu}$:
  - $G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi G T^{\mu\nu}$
  - Λ called *Cosmological constant*

General Relativity

GR Summary - 2 statements:

- Space-time is a curved pseudo-Riemannian manifold
- Matter curves space-time according to Einstein's equation:

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}}$$

- We need to a bit of mathematics to appreciate the summary
- Work in units where $c = 1$

GR

Bibliography

S. Carrol
A No-Nonsense Introduction to General Relativity[1]
http://preposterousuniverse.com/grnotes/grtinypdf.pdf

[1] Lecture loosely base on these notes.