Scattering amplitudes in maximally supersymmetric Yang-Mills theory

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Based on work in collaboration with

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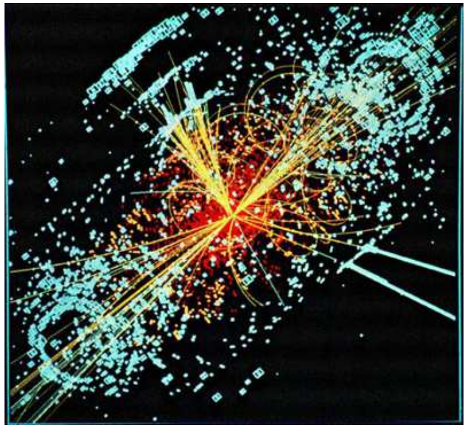
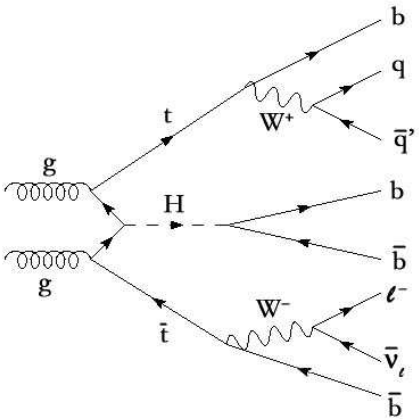
Why scattering amplitudes?

Hep-ph motivation: Search for the Higgs boson at Large Hadron Collider





- ✗ Lots of produced particles in the final state leading to large background
- ✗ Identification of Higgs boson requires detailed understanding of scattering amplitudes
- ✗ Theory should provide solid basis for a successful physics program at the LHC

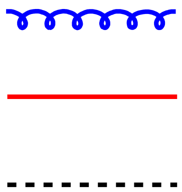
Hep-th motivation: Scattering amplitudes have a remarkable structure in planar $\mathcal{N} = 4$ SYM

- ✗ Simpler than Standard Model amplitudes but share many of the same properties
- ✗ All-order conjectures [Bern,Dixon,Smirnov], proposal for strong coupling via AdS/CFT [Alday,Maldacena]
- ✗ Hints for new symmetry – ***dual superconformal invariance*** [Drummond,Henn,GK,Sokatchev]

Maximally supersymmetric Yang-Mills theory

- ✔ Most (super)symmetric theory possible (without gravity)
- ✔ Uniquely specified by local internal symmetry group - e.g. number of colors N_c for $SU(N_c)$
- ✔ Exactly scale-invariant field theory for any coupling g^2 (Green functions are powers of distances)
- ✔ *Gluon tree amplitudes* are the same in all gauge theories

Particle content:

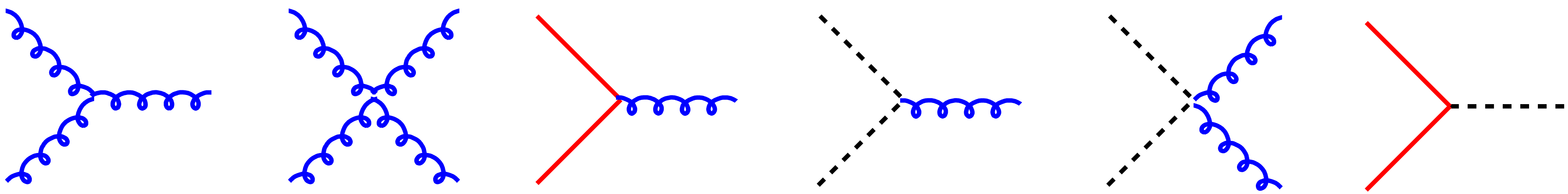


massless spin-1 gluon (= the same as in QCD)

4 massless spin-1/2 gluinos (= cousin of the quarks)

6 massless spin-0 scalars

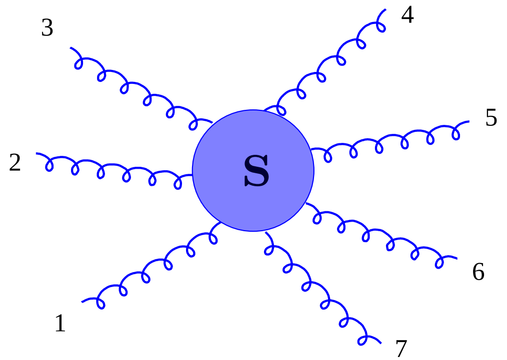
Interaction between particles:



All proportional to same dimensionless coupling g and related to each other by supersymmetry

Gluon amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory

- On-shell matrix elements of $S-$matrix:

$\mathcal{A}_n \;=$ 

- Quantum numbers of scattered gluons:

  Color: $a_i = 1, \ldots, N_c^2 - 1$

  Light-like momenta: $(p_i^\mu)^2 = 0$

  Polarization state (helicity): $h_i = \pm 1$

- Color-ordered planar gluon amplitudes:

$$\mathcal{A}_n = \text{tr}\left[T^{a_1} T^{a_2} \ldots T^{a_n}\right] A_n^{h_1, h_2, \ldots, h_n}(p_1, p_2, \ldots, p_n) + \text{[Bose symmetry]}$$

  - Supersymmetry all-loop relations:

$$A_n^{++\ldots+} = A_n^{-+\ldots+} = 0$$

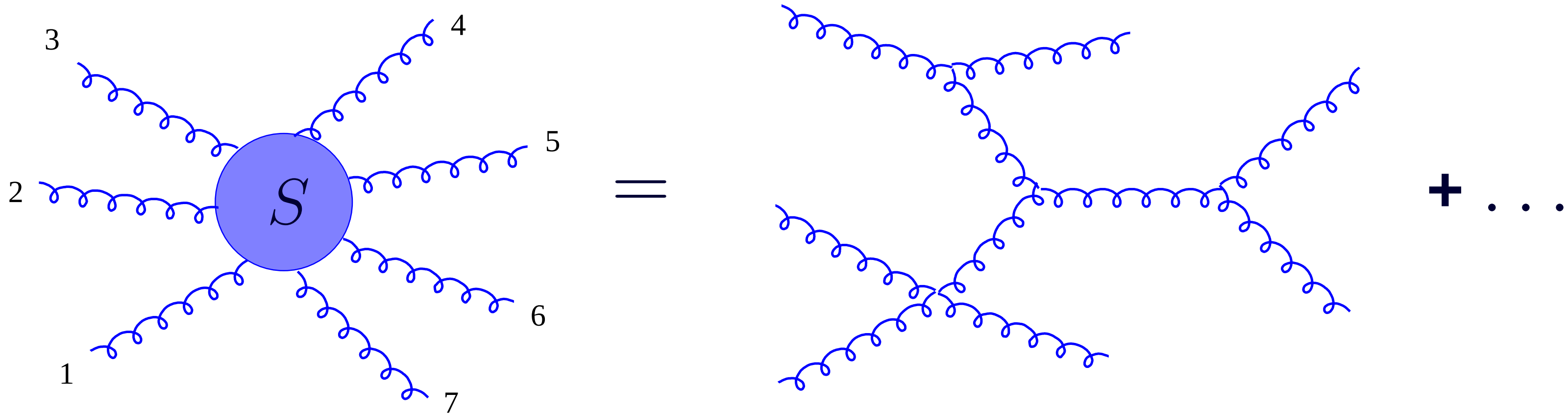
  - Classification of amplitudes according to the total helicity $h = \sum_1^n h_i$

$$\text{MHV} = \{A_n^{--+\ldots+}, A_n^{-+-\ldots+}, \ldots\}, \qquad \text{NMHV} \quad = \{A_n^{---+\ldots+}, A_n^{-+--\ldots+}, \ldots\}$$

MHV = Maximal Helicity Violating ampitudes, NMHV = next-to-MHV, ...

Hints for integrability

Gluon amplitudes at tree level:



Number of external gluons	4	5	6	7	8	9	10
Number of ‘tree’ diagrams	4	25	220	2485	34300	559405	10525900

- ✔ Number of diagrams grows factorially for large number of external gluons/number of loops
- ✔ ... but the final expression looks remarkably simple [Parke,Taylor]

$$A_n^{\mathrm{MHV,tree}}(1^-2^-3^+\dots n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle}\delta^{(4)}\Big(\sum_i p_i\Big)$$

Where does this simplicity come from ... Look for symmetries

Conformal symmetry of the amplitude

$\mathcal{N}=4$ SYM has (super)conformal $SU(2,2|4)$ symmetry to all loops:

- ✔ Consequences of conformal symmetry for the correlation functions

$$\langle O(x)O(0)\rangle \sim (x^2)^{-\Delta}$$

$$\langle O_1(x_1)O_2(x_2)O_3(0)\rangle \sim (x_{12}^2)^{\Delta_3-\Delta_1-\Delta_2}(x_2^2)^{\Delta_1-\Delta_2-\Delta_3}(x_1^2)^{\Delta_2-\Delta_3-\Delta_1}$$

- ✔ Conformal symmetry acts locally in $x-$space but non-locally in $p-$space

- ✔ Realization of conformal symmetry for the amplitudes [Witten]

$$k_{\alpha\dot\alpha}=\sum_i \frac{\partial^2}{\partial\lambda_i^\alpha\partial\tilde\lambda_i^{\dot\alpha}} \qquad \Longrightarrow \qquad k_{\alpha\dot\alpha}\mathcal{A}_n^{\mathrm{MHV}}=0$$

Can be extended to the full $SU(2,2|4)$ superconformal invariance

$$g\cdot\mathcal{A}_n^{\mathrm{MHV}}=0\,, \qquad g=\{p,\,m,\,k,\,q,\,\bar q,\,s,\,\bar s,\,\ldots\}\in SU(2,2|4)$$

Much less trivial to verify for NMHV, N^{2}MHV,... amplitudes [Bargheer et al],[GK,Sokatchev]

- ✔ *Conformal symmetry alone is not powerful enough to fix the tree amplitudes*

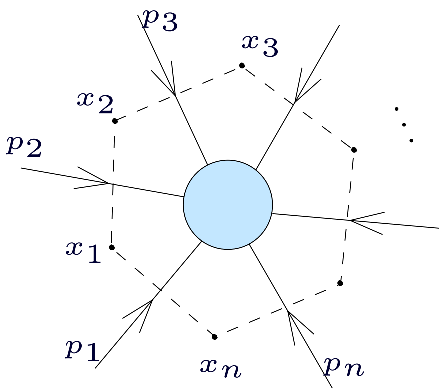
Dual $\mathcal{N} = 4$ (super)conformal symmetry

The $\mathcal{N} = 4$ amplitudes have a much bigger, dual conformal symmetry [Drummond, Henn, GK, Sokatchev]

- ✔ Examine absolute value of the amplitude:

$$\left|\hat{A}_n^{\mathrm{MHV}}\right|^2 = \frac{(S_{12})^4}{S_{12}S_{23}\dots S_{n1}}\,, \qquad (\text{with} \quad S_{ij} = (p_i + p_j)^2)$$

- ✔ Introduce dual variables (not a Fourrier transform!)



- ✗ $p_i = x_i - x_{i+1}\,, \quad x_{n+1} \equiv x_1$
- ✗ $p_i^2 = 0 \Longrightarrow (x_i - x_{i+1})^2 = 0$
- ✗ $S_{i,i+1} = (x_i - x_{i+2})^2$

- ✔ The MHV amplitude in the dual space

$$\left|\hat{A}_n^{\mathrm{MHV}}\right|^2 = \frac{[(x_1 - x_3)^2]^3}{(x_2 - x_4)^2(x_3 - x_5)^2\dots(x_n - x_1)^2}$$

Looks like $n-$point correlation function in $x-$space, *BUT x's are the momenta!*

Dual $\mathcal{N}=4$ superconformal symmetry II

- Conformal inversions in dual $x-$space

$$x_i^\mu \to \frac{x_i^\mu}{x_i^2} \qquad \Longrightarrow \qquad S_{i,i+1} \to \left(x_i^2 x_{i+2}^2\right)^{-1} S_{i,i+1}$$

Acts locally on the momenta $\Longrightarrow$ *is not related* to conformal symmetry of $\mathcal{N}=4$ SYM

- The tree-level MHV amplitude is covariant under dual conformal inversions

$$I\left[\mathcal{A}_n^{\rm MHV}\right] = \left(x_1^2 x_2^2 \ldots x_n^2\right) \times \mathcal{A}_n^{\rm MHV}$$

- Dual conformal symmetry can be extended to dual superconformal $\widetilde{SU}(2,2|4)$ symmetry

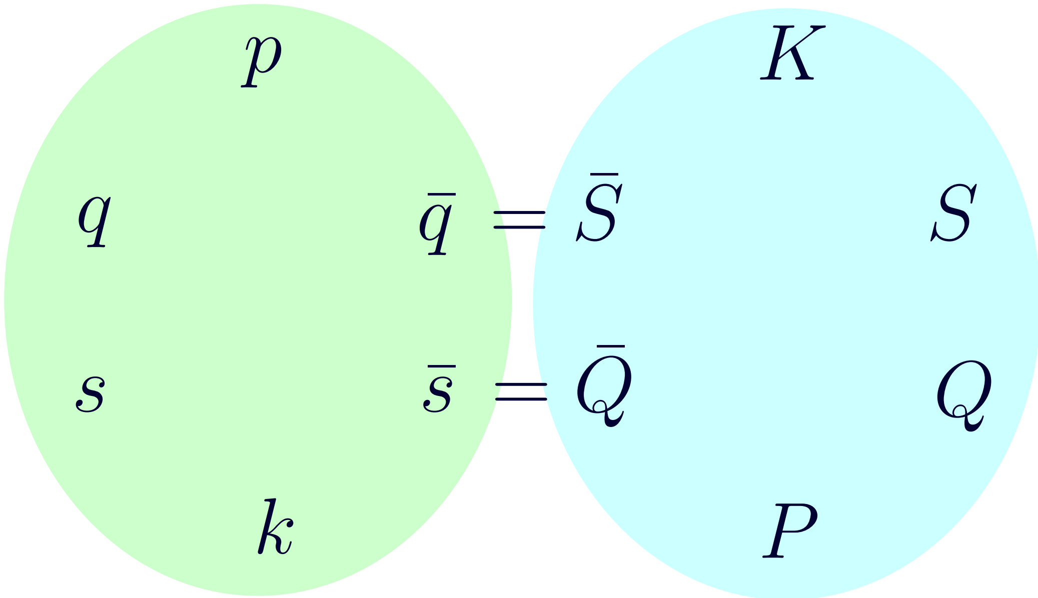
$$G \cdot \mathcal{A}_n^{\rm MHV} = 0\,, \qquad G = \{P,\, M,\, K,\, Q,\, \bar{Q},\, S,\, \bar{S},\, \ldots\} \in \widetilde{SU}(2,2|4)$$

- ***Dual superconformal symmetry is a property of all tree-level amplitudes (MHV, NMHV, N^{2}MHV,...) in $\mathcal{N}=4$ SYM theory*** [Drummond,Henn,GK,Sokatchev],[Brandhuber,Heslop,Travaglini]

Symmetries of tree amplitudes

✔ The relationship between conventional and dual superconformal $su(2,2|4)$ symmetries:

[Drummond,Henn,GK,Sokatchev]



✔ The same symmetries appear at strong coupling from invariance of $\mathrm{AdS}_5 \times \mathrm{S}^5$ sigma model under bosonic [Kallosh,Tseytlin] + fermionic T-duality [Berkovits,Maldacena],[Beisert,Ricci,Tseytlin,Wolf]

✔ (Infinite-dimensional) closure of two symmetries has Yangian structure [Drummond,Henn,Plefka]

Are tree level amplitudes completely determined by the symmetries?

Integrability of tree amplitudes

- ✔ General expression for the tree amplitude dictated by the symmetries

$$\mathcal{A}_n^{\rm N^pMHV} = \mathcal{A}_n^{\rm MHV} \times \sum_\alpha c_\alpha\, R_n^{(\alpha)} \qquad (c_\alpha \text{ arbitrary constants})$$

  - ✗ $R_n^{(\alpha)}$ are (super) invariants of ***both*** conventional (g) and dual (G) symmetries:

$$\begin{cases} g\cdot R_n = G\cdot R_n = 0 \\ R_n = \text{Polynomial in Grassmann } \theta\text{'s of degree } 4p \end{cases}$$

    General form of $R-$invariants is known [Arkani-Hamed et al],[Mason,Skinner],[Drummond,Ferro],[GK,Sokatchev]

  - ✗ c_α are fixed from *analytic properties* : $\mathcal{A}_n^{\rm N^pMHV}$ = meromorphic functions of $S_{i..j}$

- ✔ Example: $n-$particle NMHV tree amplitude [Drummond,Henn,GK,Sokatchev]

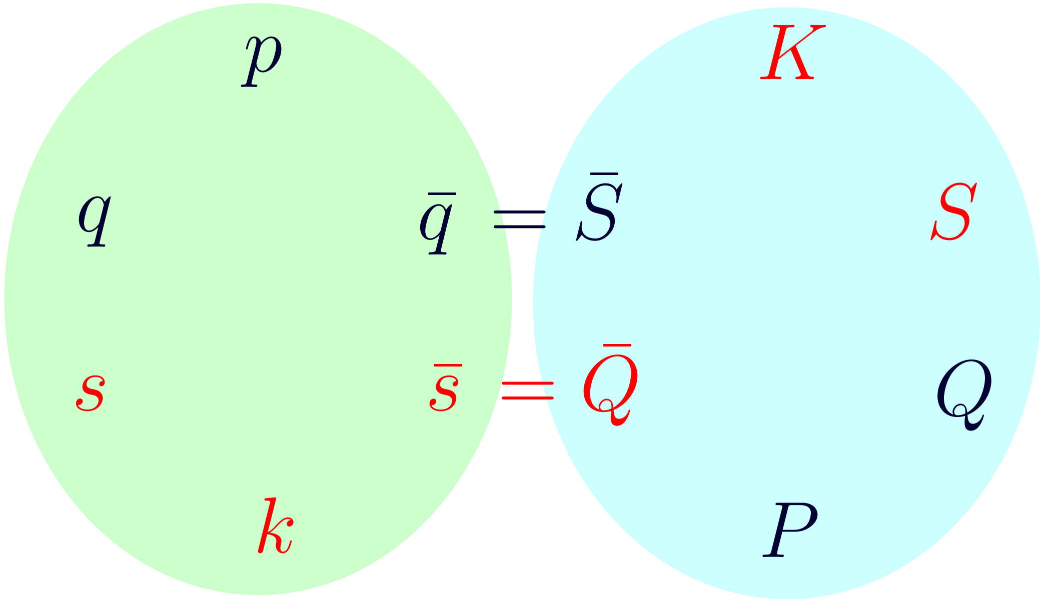
$$\mathcal{A}_n^{\rm NMHV} = \mathcal{A}_n^{\rm MHV} \sum_{4\le s+1<t\le n} R_{1st}$$

$$R_{rst} = \frac{\langle s-1 s\rangle\langle t-1 t\rangle \delta^{(4)}(\langle r|x_{rs}x_{st}|\theta_{tr}\rangle + \langle r|x_{rt}x_{ts}|\theta_{sr}\rangle)}{x_{st}^2\langle r|x_{rs}x_{st}|t-1\rangle\langle r|x_{rs}x_{st}|t\rangle\langle r|x_{rt}x_{ts}|s-1\rangle\langle r|x_{rt}x_{ts}|s\rangle}$$

All tree $\mathcal{N}=4$ amplitudes are uniquely fixed by symmetries + analyticity condition

Do symmetries survive loop corrections?

- Loop corrections to the amplitudes necessarily induce infrared divergences
- IR divergences preserve supersymmetry but break conformal + dual conformal symmetry
- Symmetries $(p, q, \bar{q}, P, Q, \bar{S}, \ldots)$ survive loop corrections, other $(s, \bar{s}, k, K, S, \bar{Q}, \ldots)$ are broken



- Dual conformal K-anomaly is *universal* for all amplitudes (MHV, NMHV,...) [Drummond,Henn,GK,Sokatchev]

$$K^{\alpha\dot{\alpha}}\, A_n \equiv \sum_{i=1}^{n}\left[2x_i^{\alpha\dot{\alpha}}(x_i\cdot\partial_{x_i}) - x_i^2\partial_{x_i}^{\alpha\dot{\alpha}}\right] A_n = \frac{1}{2}\Gamma_{\rm cusp}(g^2)\sum_{i=1}^{n} x_{i,i+1}^{\alpha\dot{\alpha}} \ln\left(\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}\right) A_n$$

The $s-$ and $\bar{Q}-$anomalies are hard to control

Dual conformal anomaly at work

Consequences of the dual conformal K-anomaly for the finite part of MHV amplitude:

✔ $n = 4, 5$ are special: the Ward identity has a *unique all-loop solution*

$$\ln A_4^{\rm MHV} = \frac{1}{4}\Gamma_{\rm cusp}(g^2)\ln^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \text{ const },$$

$$\ln A_5^{\rm MHV} = -\frac{1}{8}\Gamma_{\rm cusp}(g^2)\sum_{i=1}^{5}\ln\left(\frac{x_{i,i+2}^2}{x_{i,i+3}^2}\right)\ln\left(\frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2}\right) + \text{ const}$$

Exactly the ABDK/BDS ansatz for the 4- and 5-point MHV amplitudes!

✔ Starting from $n = 6$ there are conformal invariants in the form of cross-ratios $u_{ijkl} = \frac{x_{il}^2 x_{jk}^2}{x_{ik}^2 x_{jl}^2}$

General solution to the Ward identity contains an arbitrary function of the conformal cross-ratios

✔ The function is identified at two loops [Drummond,Henn,GK,Sokatchev] [Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich]

- ✗ Analytical expression at weak coupling [Del Duca,Durer,Smirnov],[Zhang],[Goncharov,Spradlin,Vergu,Volovich]
- ✗ Strong coupling prediction [Alday, Gaiotto,Maldacena]
- ✗ Rich structure at strong coupling (integrability, Y-system, TBA) [Alday, Gaiotto,Maldacena,Sever,Viera]

Conclusions and open questions

- ✔ Scattering amplitudes in planar $\mathcal{N}=4$ SYM possess *dual superconformal symmetry:*

  [Drummond,Henn,GK,Sokatchev]

  - ✗ Relates various particle amplitudes with different helicity configurations (MHV, NMHV,...)
  - ✗ Uniquely fixes all tree amplitudes (under appropriate analyticity conditions)
  - ✗ Imposes non-trivial constraints on the loop corrections

- ✔ The symmetry becomes manifest through *MHV amplitude/correlation function duality*

  [Alday,Eden,Korchemsky,Maldacena,Sokatchev]

Questions:

- ✗ What are integrable structures underlying scattering amplitudes in $\mathcal{N}=4$ SYM?
- ✗ 'Bethe ansatz' for all-loop amplitudes?