Holography for Schrödinger

Kostas Skenderis

University of Amsterdam

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Introduction

- Gauge/gravity dualities have been an important new tool in extracting strong coupling physics.
- The best understood examples of such dualities involve relativistic quantum field theories.
- Strongly coupled non-relativistic QFTs are common place in condensed matter physics and elsewhere.
- It is natural to wonder whether holography can be used to obtain new results about such non-relativistic strongly interacting systems.

The non-relativistic conformal group

In non-relativistic physics the Poincaré group is replaced by the Galilean group. It consist of

- the temporal translation $\mathcal{H}$, spatial translations $\mathcal{P}^i$, rotations $\mathcal{M}^{ij}$, Galilean boosts $\mathcal{K}^i$ and the mass operator $\mathcal{M}$.

The conformal extension adds to these generators

- the non-relativistic scaling operator $\mathcal{D}$ and the non-relativistic special conformal generator $\mathcal{C}$.

The scaling symmetry acts as

$$t \to \lambda^2 t, \qquad x^i \to \lambda x^i$$

Schrödinger group

- This is the maximal kinematical symmetry group of the free Schrödinger equation [Niederer (1972)], hence its name: Schrödinger group $Sch(d)$.

Interacting systems that realize this symmetry include:

- Non-relativistic particles interacting through an $1/r^2$ potential.
- Fermions at unitarity. (Fermions in three spatial dimensions with interactions fine-tuned so that the s-wave scattering saturates the unitarity bound). This system has been realized in the lab using trapped cold atoms [O'Hara et al (2002) ...] and has created enormous interest.

Holography for Schrödinger

Motivated by such applications [Son (2008)] and [K. Balasubramanian, McGreevy (2008)] initiated a discussion of holography for $(d+1)$ dimensional spacetimes with metric,

$$ds^2 = -\frac{b^2 du^2}{r^4} + \frac{2dudv + dx^i dx^i + dr^2}{r^2},$$

- When $b = 0$ this is the AdS_{d+1} metric.
- This metric realizes geometrically the Schrödinger group in $(d-1)$ dimensions.
- In order for the mass operator $\mathcal{M}$ to have discrete eigenvalue lightcone coordinate v must be compactified.

Bulk system

This metric solves the field equations for

- gravity coupled to massive vectors
- topologically massive gravity with $\mu = 3$

In the latter case the solution is called "null warped AdS_3" and it was conjectured to be dual to a $2d$ CFT with certain (c_L, c_R) [Anninos et al (2008)].

$\longrightarrow$ This is a rather different proposal for the physics of the solution.

References

- The purpose of our work is to understand how holography works for such spacetimes.
- Based on
  M. Guica, KS, M. Taylor, B. van Rees
  Holography for Schrödinger backgrounds,
  to appear

The issues

- These spacetimes are not asymptotically AdS and as such the usual set up is not automatically applicable.

Even basic issues such as

- is the dual theory a local QFT?
- what is the correspondence between bulk fields and dual operators?

are not understood.

Results

To avoid the complications of a null compact direction, we consider the spacetime with v *non-compact*.
The main result is then

- The dual theory is a deformation of a d-dimensional CFT.
- The deformation is **irrelevant** w.r.t. relativistic conformal group.
- The deformation is **exactly marginal** w.r.t. **non-relativistic** conformal group.
- The theory is non-local in the v direction.

The small b limit

In the small b limit the geometry is a small perturbation of AdS and standard AdS/CFT applies.

- Massive vector model:

$$S_{CFT} \to S_{CFT} + \int d^d x \, b^i X_i$$

$\to$ X_i has dimension $(d+1)$.
$\to$ b^i is a null vector with only non-zero component $b^v = b$.

- Topologically massive gravity:

$$S_{CFT} \to S_{CFT} + \int d^2 x \, b^{ij} X_{ij}$$

$\to$ X_{ij} has dimension $(3, 1)$.
$\to$ b^{ij} is a null tensor with only non-zero component $b^{vv} = -b^2$.

Schrödinger invariance

In both cases the non-relativistic scaling dimension of the deformation is

$$\Delta_s = d$$

We now need to understand what happens at finite b.

Finite b

Bulk perspective:

- The perturbation that solves the linearized equation around AdS, it also automatically solves the non-linear equations.

$\longrightarrow$ The theory is Schrödinger invariant for any b.

Boundary QFT perspective:

- We analyzed this question using conformal perturbation theory.

$\longrightarrow$ The deforming operator is exactly marginal.

Exact marginality

To explain this computation we need a few facts about theories with Schrödinger invariance:

- Operators are labeled by their non-relativistic scaling dimension, Δ_s and their charge under $\mathcal{M}$, the mass operator.
- In our context the mass operator is the lightcone momentum k_v.
- Operators with different k_v are considered as independent operators.
- In our case, the deforming operator has zero lightcone momentum, $k_v = 0$.

Exact marginality

To prove that the operator is exactly marginal it suffices to show that its 2-point function does not receive any corrections when we turn on b.

$$\langle X_v(k_v{=}0, u_1, x_1^i) X_v(k_v{=}0, u_2, x_2^i)\rangle_{\mathbf{b}} = \langle X_v(k_v{=}0, u_1, x_1^i) X_v(k_v{=}0, u_2, x_2^i)\rangle_{\mathbf{b=0}}$$

This can be studied using conformal perturbation theory.

Conformal perturbation theory

One can show that

$$\langle X_v(k_v) \prod_{i=1}^{n} b^\mu \cdot X_\mu(k_v{=}0) X_v(-k_v) \rangle_{\rm CFT} = \langle X_v(k_v) X_v(-k_v) \rangle_{\rm CFT} (b^v k_v)^n f(\log k_v, \dots)$$

where $f(\log k_v, \dots)$ is a dimensionless function that depends at most polynomially on $\log k_v$.

- Taking the limit $k_v \to 0$, establishes that $X_v(k_v{=}0)$ is exactly marginal.
- The dimensions of operators with $k_v \neq 0$ receive corrections,

$$\Delta_s = \Delta_s(b=0) + \sum_{n>0} \mathbf{c_n} (b k_v)^n$$

Summary

- We started with a relativistic CFT and deformed it by an irrelevant operator which is however marginal from the perspective of the Schrödinger group.
- We showed that the deformation is exactly marginal and the deformation takes the theory from a relativistic fixed point to a non-relativistic one.

Summary

- The question is then to understand the spectrum of operators in the new fixed point.
- We have seen that in the non-relativistic dimension Δ_s of operators with $k_v \neq 0$ changes as we go from one fixed point to the other.
- We will next analyze this question from the bulk perspective.

Probe scalar

Let us start by analyzing a probe scalar field in the Schrödinger background,

$$S = -\frac{1}{2} \int d^3x \sqrt{-G} \Big(\partial_\mu \Phi \partial^\mu \Phi + m^2 \Phi^2 \Big).$$

The field equations are

$$\ddot{\Phi} + 2\dot{\Phi} + \Box_\zeta \Phi - (m^2 - b^2 \partial_v^2)\Phi = 0$$

The asymptotics of the solution are

$$\Phi = e^{(\Delta_s - 2)r} \Big(\phi_{(0)}(k) + \ldots + e^{-(2\Delta_s - 2)r} \phi_{(2\Delta_s - 2)}(k) + \ldots \Big)$$

Probe scalar

- The dual operator has dimension

$$\Delta_s = 1 + \sqrt{1 + m^2 + b^2 k_v^2}$$

- For small b it takes the form we found earlier using conformal perturbation theory

$$\Delta_s = \Delta_s(b = 0) + \sum c_n (b k_v)^n$$

where $\Delta_s(b = 0) = 1 + \sqrt{1 + m^2}$ is the standard holographic formula for the dimension of a scalar operator.

Correlation functions

- To compute correlation functions we need to compute the on-shell value of the action.
- This suffers from the infinite volume divergences.
- Adapting holographic renormalization we find that we need counterterms

$$S_{\text{ct},\Delta_s \lesssim 3} = -\frac{1}{2}\int d^2k \sqrt{-\zeta}\Big((\Delta_s - 2)\Phi^2 + \frac{k_\zeta^2 \Phi^2}{2\Delta_s - 4}\Big)$$

- When $b = 0$ these reduce to the counterterms for the scalar field in AdS.

Non-locality

$$S_{\mathrm{ct},\Delta_s \lesssim 3} = -\frac{1}{2} \int d^2k \, \sqrt{-\zeta} \Big((\Delta_s - 2)\Phi^2 + \frac{k_\zeta^2 \Phi^2}{2\Delta_s - 4} \Big)$$

- Because Δ_s depends on k_v, the counterterms are not polynomials in k_v.
- The theory is non-local in the v direction.

2-point function

- Having determined the counterterms, the 2-point function can now extracted from an exact solution of the field equations.

$$\langle \mathcal{O}_{\Delta_s}(u, k_v) \mathcal{O}_{\Delta_s}(0, -k_v) \rangle = c_{\Delta_s, k_v} \delta_{\Delta, \Delta_s} u^{-\Delta_s},$$

  where c_{Δ_s, k_v} is a (specific) normalization factor.
- This is precisely of the expected form for a 2-point function of a Schrödinger invariant theory [Henkel (1993)].

Gravitational sector

We now turn to the gravitational sector and discuss the solutions to the linearized equations around the background.

- Both models (massive vector and TMG) admit two distinct sets of solutions to the linearized equations.
- The ‘T’ solutions are associated with the dual stress energy tensor,
- The ‘X’ solutions are associated with the dual deforming operator.

'X' solutions: TMG

- The mode satisfies a hypergeometric equation.
- The dimension of the dual operator is

$$\Delta_s(X_{vv}) = 1 + \sqrt{1 + b^2 k_v^2}$$

  This has the correct limit as $b \to 0$.
- The linearized solution is more singular than the Schrödinger background. This is due to the fact that the operators with $k_v \neq 0$ are irrelevant.
- The 2-point function takes the expected form.

‘X’ solutions: Massive vector

- The mode satisfies a forth order equation.
- The dimensions of the dual operators are

$$\Delta_s(X_v) = 1 + \sqrt{1 + b^2 k_v^2}; \;\; \Delta_s(X_u) = 1 + \sqrt{9 + b^2 k_v^2}$$

  This has the correct limit as $b \to 0$.
- The linearized solution is more singular than the Schrödinger background. This is due to the fact that the operators with $k_v \neq 0$ are irrelevant.
- The 2-point function takes the expected form.

'T' solutions

The metric perturbation takes the form:

$$
\begin{aligned}
h^T_{uu} &= \frac{1}{r^2} h_{(-2)uu} + \tilde{h}_{(0)uu} \log(r^2) + h_{(0)uu} + r^2 h_{(2)uu} \\
h^T_{uv} &= \frac{1}{r^2} h_{(-2)uv} + \tilde{h}_{(0)uv} \log(r^2) + h_{(0)uv} + r^2 h_{(2)uv} \\
h^T_{vv} &= h_{(0)vv} + r^2 h_{(2)vv},
\end{aligned}
$$

- These modes at $b = 0$ reduce to the modes that couple to the energy momentum tensor, T_{ij}.
- The solution is more singular than the Schrödinger background. This is related with the fact that $\Delta_s(T_{uu}) = 4$ and thus this operator is irrelevant (from the perspective of Schrödinger).

Stress energy tensor

- There are subtleties in understanding this sector.
- In a **non-relativistic theory** the tensor that contains the conserved energy and momentum is not symmetric and therefore cannot couple to any metric mode. This tensor couples instead to the vielbein.
- Still in progress ...

Conclusions

We have argued that the dual to the Schrödinger backgrounds is

- a deformation of a d-dimensional CFT.
- The deformation is **irrelevant** w.r.t. relativistic conformal group.
- The deformation is **exactly marginal** w.r.t. **non-relativistic** conformal group.
- The theory is non-local in the v direction.

Null dipole theory

It was argued in [Maldacena et al (2008)] using TsT transformations that the massive vector theory in $d = 4$ is dual to the null dipole theory, a non-local deformation of $N = 4$ SYM.

In the null dipole theory, the ordinary product is replaced by a non-commutative product that depends on a null vector [Ganor et al (2000)]. Expressed in terms of ordinary products the null dipole theory contains terms that

- irrelevant from the relativistic CFT point of view
- marginal from the Schrödinger perspective

$\longrightarrow$ This is in exact agreement with our findings.

Outlook

- Very little is currently known about the null dipole theories.
  - Is the structure of divergences the same as what we found in gravity?
  - Does the dipole theory resum the series in b to produce the surd?
- Understand better the stress energy sector.