DIFFERENTIAL REDUCTION TECHNIQUES FOR THE EVALUATION OF FEYNMAN DIAGRAMS

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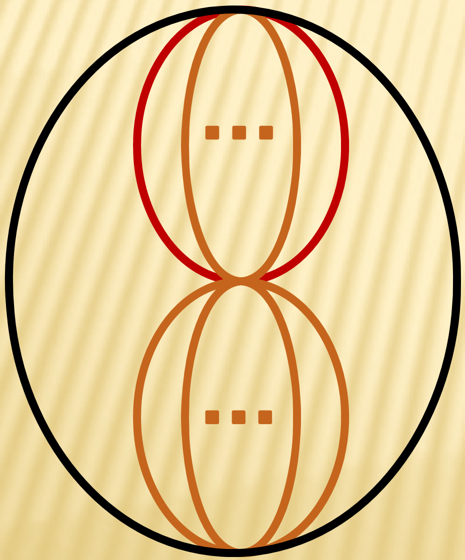


Stable reduction reduction methods will be important in the evaluation of high-order perturbative diagrams appearing in QCD and mixed QCD-electroweak radiative corrections at the LHC. We describe differential reduction techniques in the hypergeometric function representation of Feynman diagrams and present some representative examples.

HYPERGEOMETRIC FUNCTION APPROACH

- Regge proposed (about 45 years ago) that Feynman diagrams could be represented in terms of hypergeometric functions.
- This proposal was based on a study of the singularities of Feynman diagrams as a function complex momenta (Landau singularities). Matching the of the HG function to the diagram would determine the appropriate representation.
- Much work has been done on finding the representation of various diagrams in terms of HG functions, and finding recursion relations among them which can be the basis for a reduction algorithm.

HYPERGEOMETRIC SERIES

A Laurent series in r variables

$$\Phi(\vec{x}) = \sum C(\vec{m})\, x_1^{m_1} \cdots x_r^{m_r}$$

is hypergeometric if for each i, the ratio $C(\vec{m}+\vec{e}_i)/C(\vec{m})$ is a rational function in the multi-index $\vec{m}$, with

$$\vec{e}_i = (0,\cdots,0,1,0,\ldots 0)$$

i^{th} place

This type of HG series called a Horn series.

HORN-TYPE HYPERGEOMETRIC FUNCTIONS

- In general, starting with the Feynman parameterization, any Feynman diagram containing arbitrary powers of propagators of the form $(k^2 - m^2)^{-j}$ can be written in terms of a multiple Mellin-Barnes integral leading to a linear combination

$$\sum_{\vec{\alpha}} C_{\vec{\alpha}} x_1^{\alpha_1} \cdots x_r^{\alpha_r} \; \Phi\left(\vec{A}; \vec{B}; \vec{x}\right)$$

of Horn-type hypergeometric functions, where x_j are rational functions of masses and momenta, α_j depends on the powers of propagators and dimension of space-time, and C's are ratios of Γ functions with arguments depending on the α's.

HORN-TYPE HYPERGEOMETRIC FUNCTIONS

Specifically,

$$\Phi\left(\vec{A};\vec{B};\vec{x}\right)=\sum_{\vec{m}=\vec{0}}^{\infty}\left(\frac{\prod_{j=1}^{K}\Gamma\left(\sum_{a=1}^{r}\mu_{ja}m_a+A_j\right)}{\prod_{k=1}^{L}\Gamma\left(\sum_{b=1}^{r}\nu_{kb}m_b+B_k\right)}\right)x_1^{m_1}\cdots x_r^{m_r}$$

with μ_{ja}, ν_{kb} rational, A_j, B_k complex.

An important property of Horn-type hypergeometric functions is the existence of a set of differential contiguous relations between functions with shifted arguments.

DIFFERENTIAL REPRESENTATION

The Horn-Type HG series can be shown to satisfy a system of differential equations of the form

$$Q_j\left(\sum_{k=1}^{r} x_k \frac{\partial}{\partial x_k}\right)\frac{\Phi(\vec{x})}{x_j} = P_j\left(\sum_{k=1}^{r} x_k \frac{\partial}{\partial x_k}\right)\Phi(\vec{x}), \qquad j = 1, \cdots, r$$

with polynomials P_j, Q_r satisfying

$$\frac{C(\vec{m}+\vec{e}_j)}{C(\vec{m})} = \frac{P_j(\vec{m})}{Q_j(\vec{m})}$$

DIFFERENTIAL CONTIGUOUS RELATIONS

- Both the upper (A) and lower (B) arguments can be shifted by applying differential operators:

$$\Phi\left(\vec{A}+\vec{e}_c;\vec{B};\vec{x}\right)=\left(\sum_{b=1}^{r}\mu_{ca}x_a\frac{\partial}{\partial x_a}+A_c\right)\Phi\left(\vec{A};\vec{B};\vec{x}\right)=U^{+}_{[A_c\to A_c+1]}\Phi\left(\vec{A};\vec{B};\vec{x}\right)$$

$$\Phi\left(\vec{A};\vec{B}-\vec{e}_c;\vec{x}\right)=\left(\sum_{b=1}^{r}\nu_{cb}x_b\frac{\partial}{\partial x_b}+B_c\right)\Phi\left(\vec{A};\vec{B};\vec{x}\right)=L^{-}_{[B_c\to B_c-1]}\Phi\left(\vec{A};\vec{B};\vec{x}\right)$$

- If inverse operators $U^{-}_{[A_c\to A_c-1]}$, $L^{+}_{[B_c\to B_c+1]}$ can be found, they can be applied to form the basis of a reduction algorithm relating all HG functions related by integer shifts in the arguments to a single HG function.

TAKAYAMA ALGORITHM

- The complete differential reduction for ${}_2F_1$ was constructed by Gauss (1823). The inverse operators for the general Horn-type functions can be constructed by the Takayama algorithm.
  [Nobuki Takayama, Japan J. Appl. Math 6 (1989) 147].
- The functions $\Phi\left(\vec{A};\vec{B};\vec{x}\right)$ satisfy differential equations $D_j\Phi\left(\vec{A};\vec{B};\vec{x}\right)=0, \quad j=1,\ldots,r$
  with

$$D_j = Q_j\left(\sum_{k=1}^{r} x_k \frac{\partial}{\partial x_k}\right)\frac{1}{x_j} - P_j\left(\sum_{k=1}^{r} x_k \frac{\partial}{\partial x_k}\right).$$

TAKAYAMA ALGORITHM

- Let $\mathfrak{D}$ be the ring of differential operators with rational functions of $\vec{x}$ as coefficients. Let $\mathfrak{I}$ be the left ideal in $\mathfrak{D}$ of generated by the differential operators D_j and construct a Gröbner basis $\mathfrak{G} = \{G_i \mid i = 1, \ldots, q\}$ of $\mathfrak{I}$.
- Then $U^-_{[A_c \to A_c - 1]}$, $L^+_{[B_c \to B_c + 1]}$ are solutions to the linear equations

$$\sum_{i=1}^{q} f_i(\vec{x}) G_i + U^-_{[A_c+1 \to A_c]} U^+_{[A_c \to A_c+1]} = 1,$$

$$\sum_{i=1}^{q} g_i(\vec{x}) G_i + L^+_{[B_c-1 \to B_c]} L^-_{[B_c \to B_c-1]} = 1.$$

where f_i, g_i are arbitrary rational functions. Solutions exist if the left ideal generated by $\mathfrak{G} \cup \{U^+_{\gamma_c}\}$ [or $\{L^-_{\sigma_c}\}$] spans $\mathfrak{D}$.

DIFFERENTIAL REDUCTION

Once the raising and lowering operators are available, it is possible to express all HG functions with integer shifts in terms of an original function $\Phi\left(\vec{A};\vec{B};\vec{x}\right)$ and polynomials such that $P_0(\vec{x}), P_{m_1,\cdots,m_r}(\vec{x})$

$$P_0(\vec{x})\Phi\left(\vec{A}+\vec{a};\vec{B}+\vec{b};\vec{x}\right)=\sum_{m_1,\ldots,m_r} P_{m_1,\ldots,m_r}(\vec{x})\prod_{i=1}^{r}\left(\frac{\partial}{\partial x_i}\right)^{m_i}\Phi\left(\vec{A};\vec{B};\vec{x}\right)$$

Cases where $x_i = x_j$ or $P_0(\vec{x})=0$ require a limiting procedure to define the reduction.

GENERALIZED HYPERGEOMETRIC FUNCTIONS

Generalized HG Functions have the form

$$ {}_pF_{p-1}(\vec{a};\vec{b};z) = {}_pF_{p-1}\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_{p-1} \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{p-1}(b_j)_n} \frac{z^n}{n!} $$

with the Pochhammer symbol $(a)_n = \Gamma(a+n)/\Gamma(a)$

- They satisfy a differential equation

$$ \left[z\prod_{i=1}^{p}\left(z\frac{d}{dz}+a_i\right) - z\frac{d}{dz}\prod_{k=1}^{p-1}\left(z\frac{d}{dz}+b_k-1\right)\right] {}_pF_{p-1}(\vec{a};\vec{b};z) = 0 $$

- The raising and lowering operators are

$$ U^{+}_{[a_i \to a_i+1]} = 1 + \frac{z}{a_i}\frac{d}{dz}, \qquad L^{-}_{[b_i+1 \to b_i]} = 1 + \frac{z}{b_i-1}\frac{d}{dz}. $$

RESULT OF REDUCTION

- This allows a given HG function ${}_{p+1}F_p(\vec{a}+\vec{m},\vec{b}+\vec{n};z)$ to be expressed in terms of a basic function ${}_{p+1}F_p(\vec{a},\vec{b};z)$ and p derivatives:

$$S(\vec{a},\vec{b},z)\ {}_{p+1}F_p(\vec{a}+\vec{m},\vec{b}+\vec{n};z)$$
$$=\sum_{k=0}^{p} R_k(\vec{a},\vec{b},z)\left(z\partial_z\right)^k\ {}_{p+1}F_p(\vec{a},\vec{b};z)\,.$$

  where S, R_k are polynomials in all parameters.

- A program HYPERDIRE has been written to automate differential reduction.

  V.V. Bytev, M. Kalmykov, and B. Kniehl, in preparation. See Nucl. Phys. B836[FS] (2010) 129 for the theory and some examples which follow.

CRITERIA FOR REDUCIBILITY

For certain cases of the parameters, the r.h.s. is further reducible: can be expressed in terms of lower-order HG functions or with fewer derivatives.

I: If one of the a_i is an integer, only $p-1$ derivatives are needed: the p^{th} term becomes a polynomial.

II: If one of the differences $a_i - b_i$ is a positive integer (or 0) and certain conditions hold for the a_i, the r.h.s. can be expressed in terms of lower-order HG functions.

III: If at least two of the differences $a_i - b_i - 1$ are positive integers, and certain conditions hold on the a_i, the r.hs. can be expressed in terms of lower-order HG functions.

IV: ${}_{p+1}F_p\left(\vec{A}+\vec{m}, \vec{a}+\vec{k}, \vec{A}+\vec{m}+\vec{1}, \vec{b}+\vec{l}; z\right)$ with integers $\vec{m}, \vec{k}, \vec{l}$ can be expressed in terms of ${}_{p+1}F_p\left(\vec{A}, \vec{a}, \vec{A}+\vec{1}, \vec{b}; z\right)$ HG functions of lower order, and derivatives.

EXAMPLE: SUNSET DIAGRAMS

The q-loop sunset diagram with 2 lines of mass m and $q-1$ massless lines is

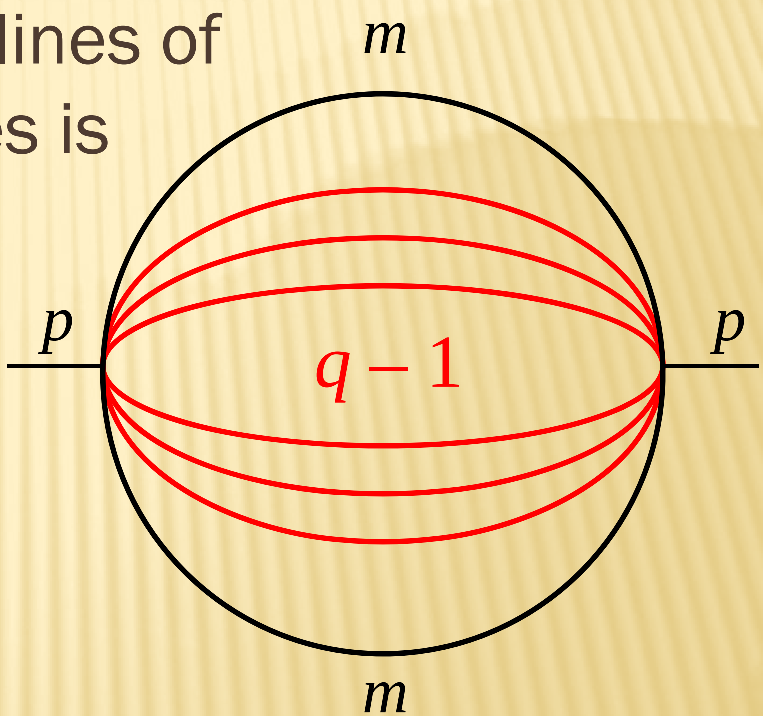
$$J_{22}^{q}\left(m^2, p^2, \alpha_1, \alpha_2, \sigma_1, \cdots, \sigma_{q-1}\right)$$

with massive denominators

$$\left(k_q^2 - m^2\right)^{\alpha_1}, \quad \left[\left(p + \sum k_i\right)^2 - m^2\right]^{\alpha_1}$$

and massless denominators

$$\left(k_1^2\right)^{\sigma_1}, \cdots, \left(k_{q-1}^2\right)^{\sigma_{q-1}} .$$



The Mellin-Barnes representation leads to

$${}_4F_3\left(\left.\begin{matrix} \alpha_1 + \sigma - \frac{n}{2}(q-1), \alpha_2 + \sigma - \frac{n}{2}(q-1), \sigma - \frac{n}{2}(q-2), \alpha + \sigma - \frac{n}{2}q \\ \frac{n}{2}, \sigma + \frac{1}{2}\left(\alpha - n(q-1)\right), \sigma + \frac{1}{2}\left(\alpha + 1 - n(q-1)\right) \end{matrix}\right| \frac{p^2}{4m^2}\right)$$

where α, σ are the sums of the two kinds of exponent.

EXAMPLE: SUNSET DIAGRAMS

The one-loop case can be further reduced:

For $q = 1$ ($\sigma = 0$) and integer a_i , the hypergeometric function is reducible via Criterion II. The $n/2$ upper and lower parameters can be removed:

$$ {}_4F_3\left(\left. \begin{matrix} \alpha_1, \alpha_2, \frac{n}{2}, \alpha - \frac{n}{2} \\ \frac{n}{2}, \frac{\alpha}{2}, \frac{\alpha+1}{2} \end{matrix} \right| \frac{p^2}{4m^2} \right) = {}_3F_2\left(\left. \begin{matrix} \alpha_1, \alpha_2, \alpha - \frac{n}{2} \\ \frac{\alpha}{2}, \frac{\alpha+1}{2} \end{matrix} \right| \frac{p^2}{4m^2} \right) $$

Compare Boos & Davydychev, Theor. Math. Phys. 89 (1991) 1052

This still satisfies Criterion II, since for even α, either $\alpha_1 - \alpha/2$ or $\alpha_2 - \alpha/2$ must be a positive integer or 0, while for odd α, similar reasoning applies to $(\alpha + 1)/2$.

Thus, we can reduce the result to ${}_2F_1$ with one integer upper parameter.

EXAMPLE: SUNSET DIAGRAMS

The two-loop case can also be further reduced:

For $q = 2$ and σ, a_i integers, the parameters become

$$ {}_4F_3\left(\begin{matrix} \alpha_1 + \sigma - \frac{n}{2}, \alpha_2 + \sigma - \frac{n}{2}, \sigma, \alpha + \sigma - n \\ \frac{n}{2}, \sigma + \frac{1}{2}(\alpha - n), \sigma + \frac{1}{2}(\alpha + 1 - n) \end{matrix} \middle| \frac{p^2}{4m^2} \right) $$

which has integer parameter differences and an integer upper parameter, so it can be reduced to ${}_3F_2$ and its first derivative, plus a rational function, with ${}_3F_2$ of the form

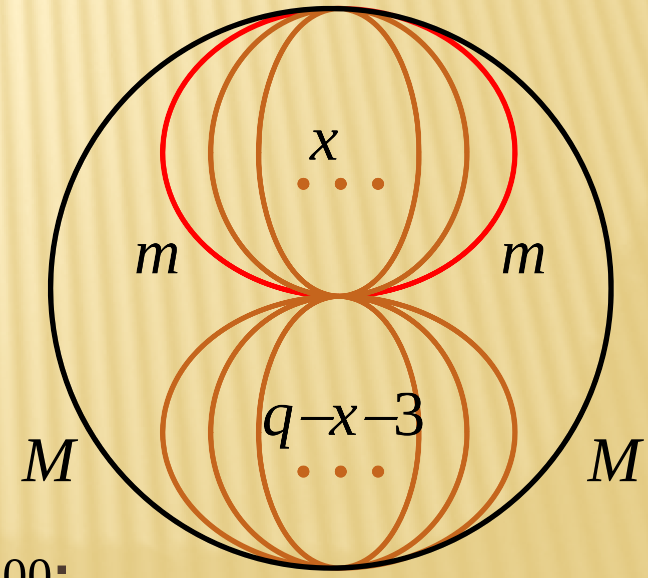
$$ {}_3F_2\left(\begin{matrix} 1, I_1 - \frac{n}{2}, I_2 - n \\ I_3 + \frac{n}{2}, I_4 - \frac{n-1}{2} \end{matrix} \middle| \frac{p^2}{4m^2} \right). $$

EXAMPLE: BUBBLE DIAGRAM

Consider the q-loop vacuum bubbles where the two black lines have mass M, the two red lines have mass m, and the $q-3$ gold lines are all massless.

The propagators of mass m have exponents α_1, α_2, the propagators of mass M have exponents β_1, β_2, the x upper massless propagators have exponents σ_i, and the $q-x-3$ lower ones have exponents ρ_i.

This Feynman diagram is denoted $B^q{}_{112200}$.



EXAMPLE: BUBBLE DIAGRAM

In this case, a lengthy expression is obtained giving a sum of four HG functions ${}_7F_6$. These can be reduced to

$$(z\partial_z)^k \;{}_4F_3\left(\begin{matrix} I_1 - \frac{n}{2}(x-1), I_2 - \frac{n}{2}x, I_3 - \frac{n}{2}(x+1), I_4 + \frac{1}{2} + \frac{n}{2}(q-x-2) \\ I_5 + \frac{n}{2}, I_6 + \frac{n}{2}(q-x-1), I_7 + \frac{1}{2} - \frac{n}{2}x \end{matrix} \middle| \frac{M^2}{m^2}\right)$$

$$(z\partial_z)^k \;{}_4F_3\left(\begin{matrix} I_1 - \frac{n-1}{2}, I_2 - \frac{n}{2}(q-2), I_3 - \frac{n}{2}(q-1), I_4 - \frac{n}{2}q \\ I_5 + \frac{n}{2}(q-x-1), I_6 + \frac{n}{2}(q-x-2), I_7 + \frac{1}{2} - \frac{n}{2}(q-1) \end{matrix} \middle| \frac{M^2}{m^2}\right)$$

$$(z\partial_z)^k \;{}_4F_3\left(\begin{matrix} 1, I_1 + \frac{1}{2}, I_3 - \frac{n}{2}(q-1), I_2 - \frac{n}{2}(q-2), I_4 - \frac{n}{2}(q-3) \\ I_5 + \frac{n}{2}, I_6 + \frac{1}{2} - \frac{n}{2}(q-2), I_7 - \frac{n}{2}(q-x-2), I_8 - \frac{n}{2}(q-x-3) \end{matrix} \middle| \frac{M^2}{m^2}\right)$$

for $k = 0, 1, 2, 3$ and I_i integers.

EXAMPLE: BUBBLE DIAGRAM

- In the special case $x = 0$, the first of these HG functions can be further reduced to

$$(z\partial_z)^k \; {}_3F_2\left(\begin{matrix} 1, I_2 - \frac{n}{2}, I_3 + \frac{1}{2} + \frac{n}{2}(q-2) \\ I_4 + \frac{n}{2}(q-1), I_5 + \frac{1}{2} \end{matrix} \middle| \frac{M^2}{m^2}\right)$$

for $k = 0, 1$ and I_i integers.

- In the special case $x = 1$, the first of these HG functions can be reduced to

$$(z\partial_z)^k \; {}_4F_3\left(\begin{matrix} 1, I_1 - \frac{n}{2}, I_2 - n, I_3 + \frac{1}{2} + \frac{n}{2}(q-3) \\ I_4 + \frac{n}{2}, \; I_5 + \frac{n}{2}(q-2), I_5 - \frac{n-1}{2} \end{matrix} \middle| \frac{M^2}{m^2}\right)$$

for $k = 0, 1, 2$ and I_i integers.

ENUMERATING MASTER INTEGRALS

All examples we considered give a Feynman diagram of the form

$$\Phi\left(n,\vec{j};z\right)=\sum_{i=1}^{k} z^{l_i} C_{l_i}\left(n,\vec{j}\right)\ {}_{p+1}F_p\left(\begin{matrix}\vec{A}_i\\ \vec{B}_i\end{matrix}\middle|\,\kappa\, z\right)$$

where $\vec{j}$ is a list of powers of propagators, n is the dimension, and z is a ratio of kinematic parameters, while κ are rational numbers, and C_{l_i} are products of Γ functions depending only on n and $\vec{j}$.

The number of basis elements in the differential reduction is the highest power v of the differential operator in the expansion

$${}_{p+1}F_p\left(\begin{matrix}\vec{A}\\ \vec{B}\end{matrix}\middle|\, z\right)=\sum_{l=1}^{v} R_l(z)\left(z\partial_z\right)^l\ {}_{s+1}F_s\left(\begin{matrix}\vec{A}-\vec{I}_1\\ \vec{B}-\vec{I}_2\end{matrix}\middle|\, z\right).$$

Where R_l are rational functions and $\vec{I}_i$ are lists of integers.

ENUMERATING MASTER INTEGRALS

- The Feynman diagram $\Phi(z)$ can alternatively be expressed in terms of a set of master integrals $\Phi_k(z)$ that may be derived from $\Phi(z)$ via integration by parts (IBP), symbolically

$$\Phi(n, \vec{j}; z) = \sum_{k=1}^{h} B_k(n, \vec{j}; z)\, \Phi_k(n; z)$$

where terms expressible solely in terms of gamma functions are not counted.

- The number of terms in this expansion is related to the number of derivatives in the differential reduction:

$$h = v + 1.$$

This is independent of the number k of hypergeometric functions in the original expression.