Constraining CFTs

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- with **R.Rattazzi, E.Tonni, A.Vichi** 0807.0004
- with **A.Vichi** 0905.2211
- with **F. Caracciolo** 0912.2726

Toy Problem

- Unitary CFT in D=4 + Z_2 symmetry

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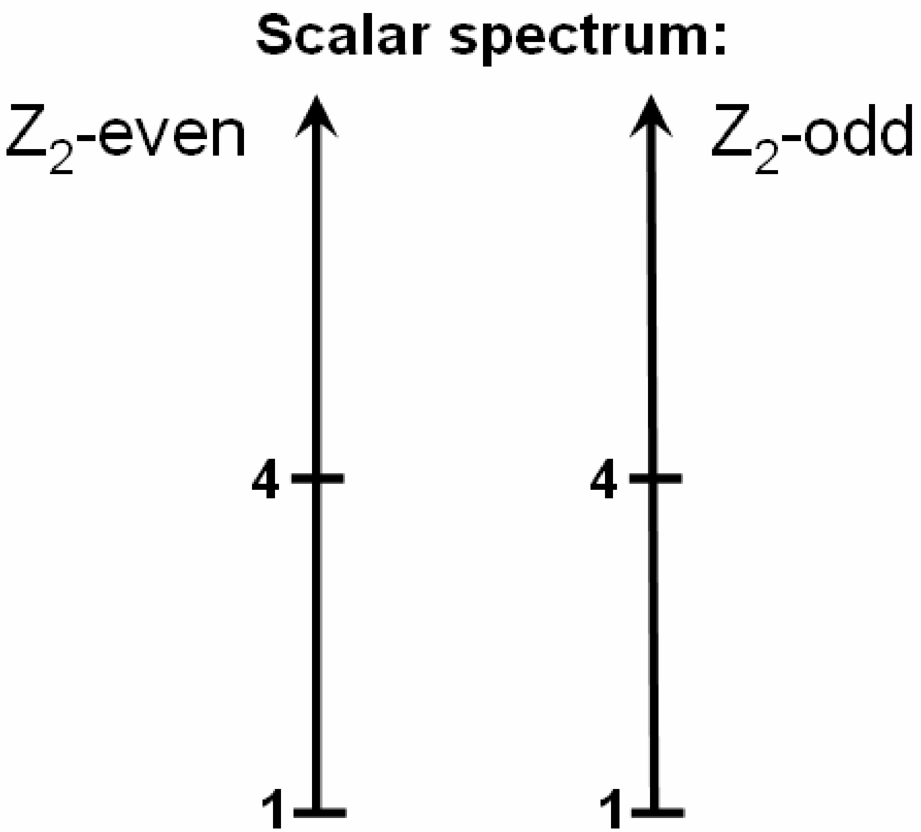

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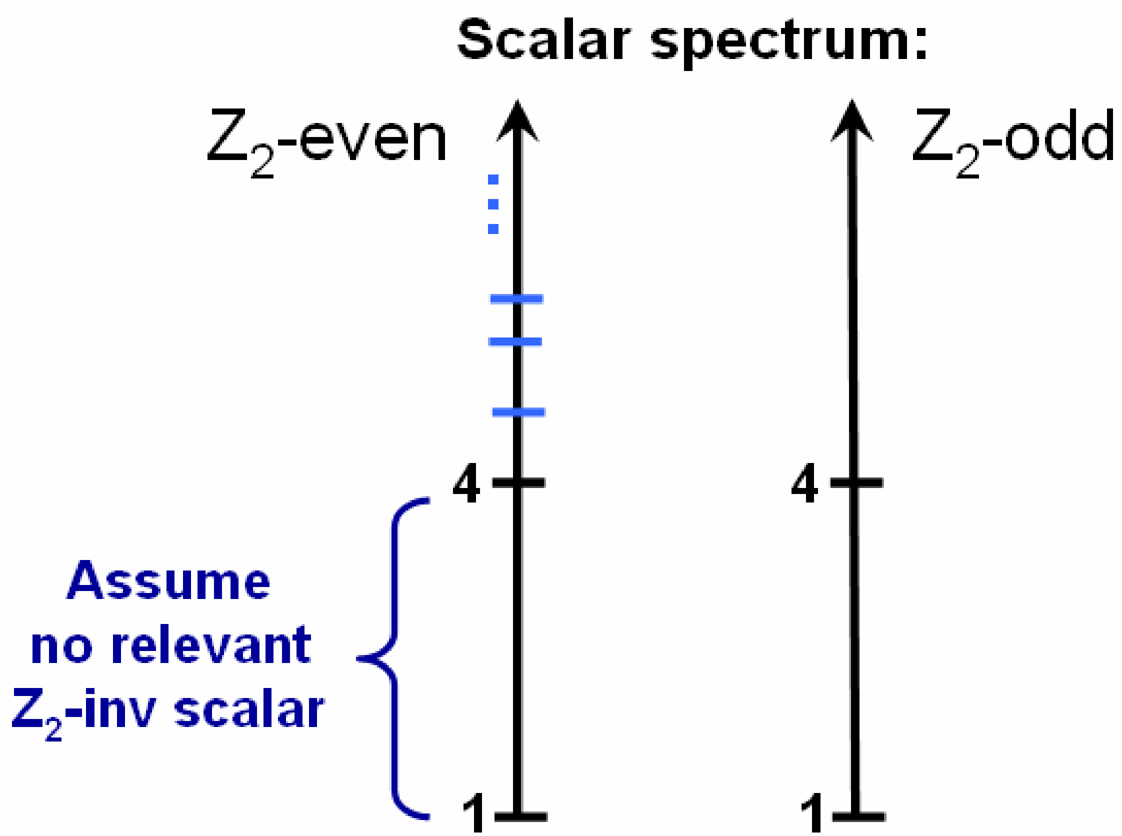

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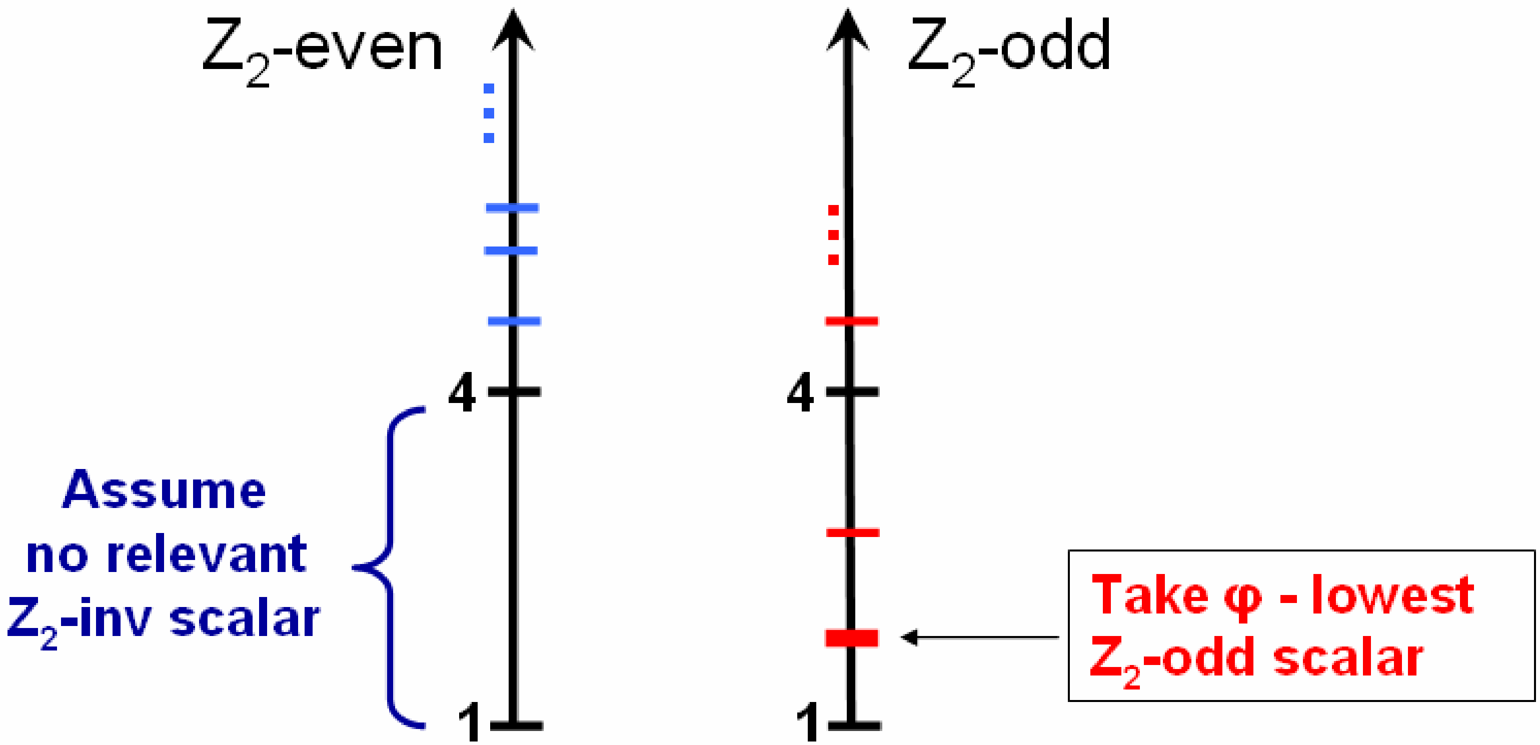
Scalar spectrum:



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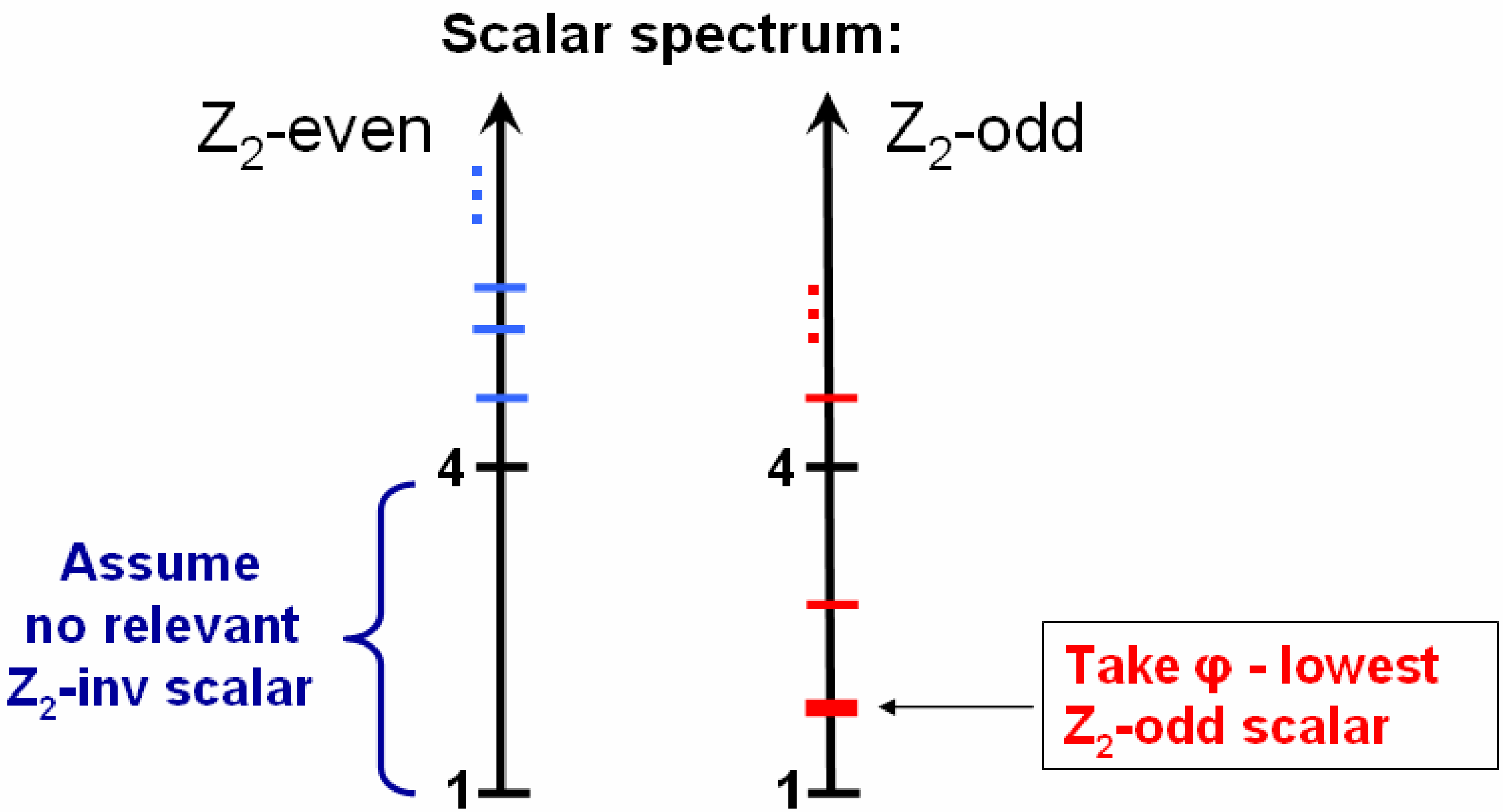

Question: What is minimal possible dim(φ)?

(assuming no relevant Z_2-inv scalar)

Why expect dim(φ) $\rightarrow$1 is impossible

Consider OPE: $\varphi \times \varphi \supset$ "φ^2" - Z_2-even

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$$\dim(\text{“}\varphi^2\text{”}) \to 2$$

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How to make this argument rigorous?

Classic theorem that $\dim(\varphi)=1$ **field is free** does not help;
Standard proof uses $\partial^2\varphi=0$; Does NOT generalize to $\dim(\varphi)=1+\varepsilon$

Real problem

Consider a QCD-like theory: $\mathcal{L} = \mathbf{Tr}\, F_{\mu\nu}^2 + \bar{\Psi} D_\mu \Psi$

As. free for $N_f < 5.5 N_c$

Expect 'conformal window' for $N_f \to 5.5 N_c$

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- Global symmetry group $G = SU(N_f)_L \times SU(N_f)_R$
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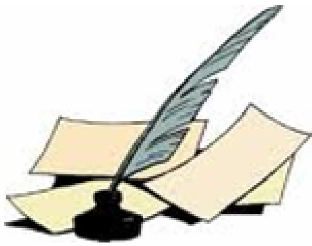
Why care:

Such operators could play a role of 'composite' Higgs field in Technicolor-like UV-completions of the Standard Model.

$\dim \text{“}\bar{\Psi}\Psi\text{”} \to 1$ would be best. **Lower bound?**

Literature



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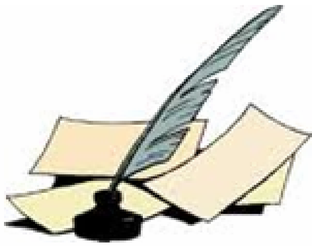

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Hellerman 0902.2790,
Hellerman, Schmidt-Colinet 1007.0756

similar problems in 2D using modular invariance

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Gaberdiel, Gukov, Keller, Moore, Ooguri 0805.4216

2D under extra assumptions (SUSY, holo factorization)

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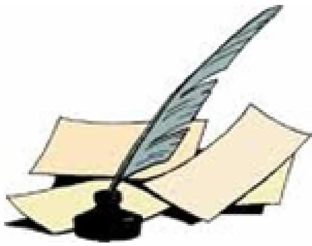

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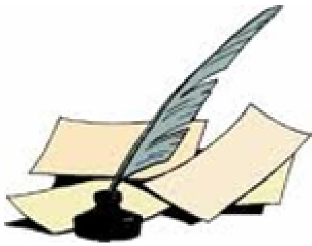

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Heemskerk, Penedones, Polchinski, Sully 0907.0151
Heemskerk, Sully 1006.0976

'Holography from CFT'

Preliminary idea

2-point
3-point $\Rightarrow$ CFT kinematics

CFT *dynamics* begins at 4-point

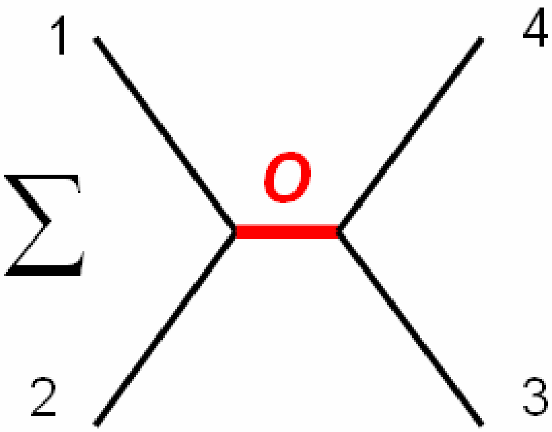
What goes wrong with

$$< \phi\phi\phi\phi >$$

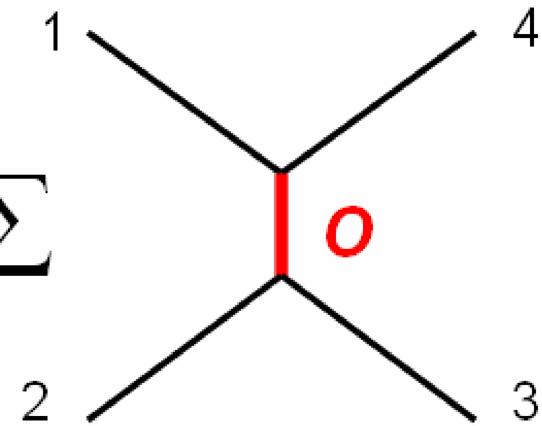
when dim $\varphi \to 1$ but dim$(\varphi^2) > 4$?

Crossing symmetry

$< \phi\phi\phi\phi > = \sum$



$= \sum$



‘Bootstrap equation’

Polyakov 1974
Belavin Polyakov Zamolodchikov 1984

Can pull out something?

Preparation 1: **OPE**

$$\varphi(x)\varphi(0) \sim \frac{1}{|x|^{2d}} \sum c_{\Delta,l} \, |x|^{\Delta} \, O_{\Delta,l}(0) + \textbf{descendants}$$

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2) lower bounds on operator dimensions:

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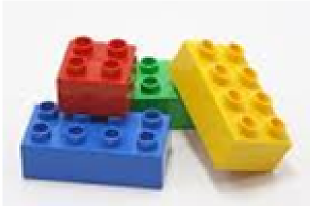
BUT: not immediately useful for imposing crossing symmetry

Preparation 2: Conformal Block Decomposition



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$$d = \dim \phi$$

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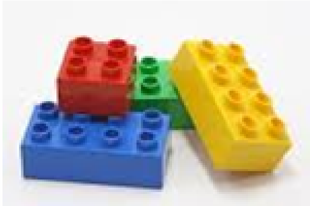

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$$k_\beta(z) = z^{\beta/2+1} {}_2F_1(\beta/2, \beta/2, \beta, z)$$

Dolan, Osborn 2001

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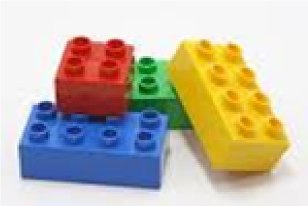

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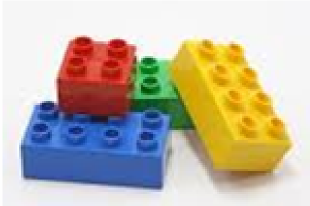

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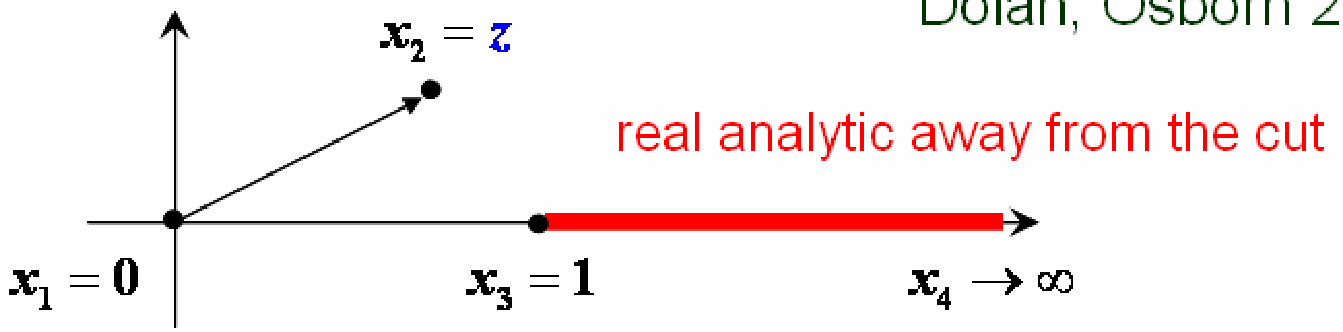

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Crossing + CB = Sum rule

$$v^d G(u,v) = u^d G(v,u)$$

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Sum rule:

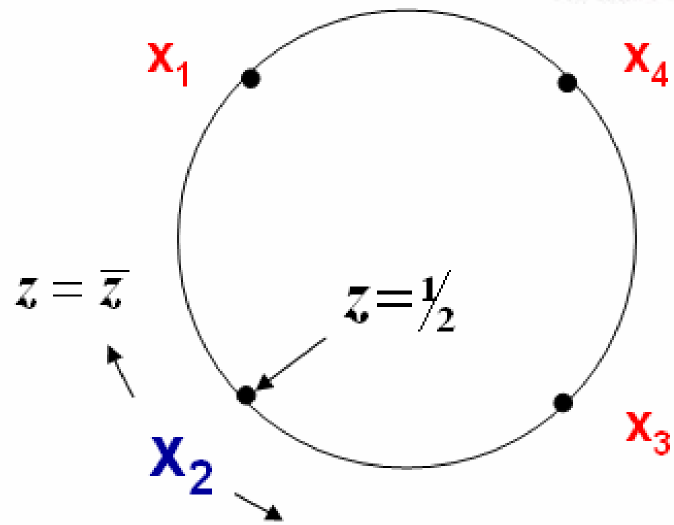
$$1 = \sum (c_{\Delta,l})^2 F_{d,\Delta,l}(u,v)$$

$$F = \frac{v^d \mathbf{CB}_{\Delta,l}(u,v) - u^d \mathbf{CB}_{\Delta,l}(v,u)}{u^d - v^d}$$

Functional equation involving **squares** of OPE coefficients

Solving toy problem

Vary x_2 near z=1/2
(4 points in vertices of a square)



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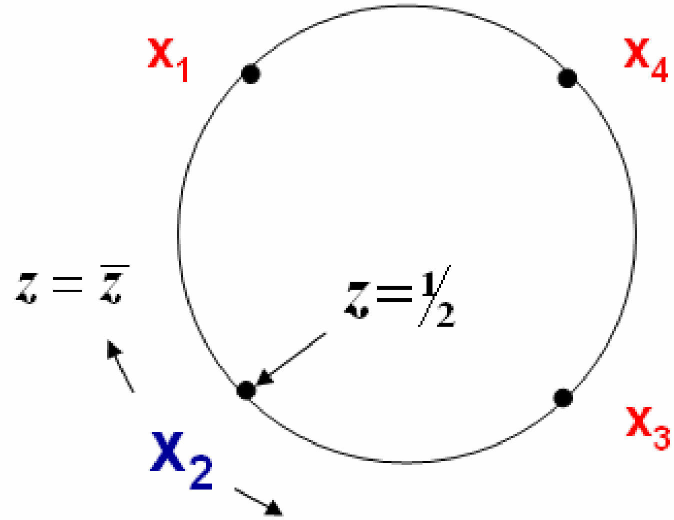
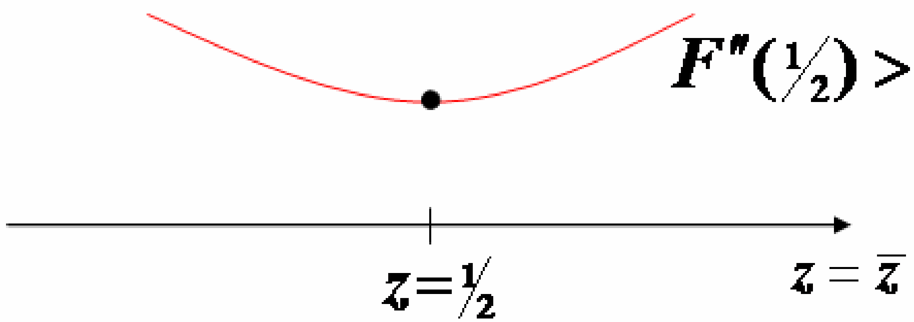


Fix d~1 and study behavior of different terms:



$F''(1/2) > 0$ for all

$\Delta \geq 4$	$(l=0)$
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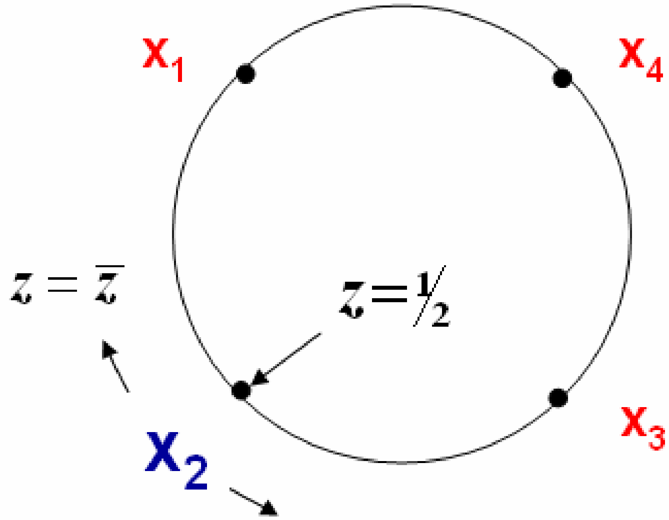

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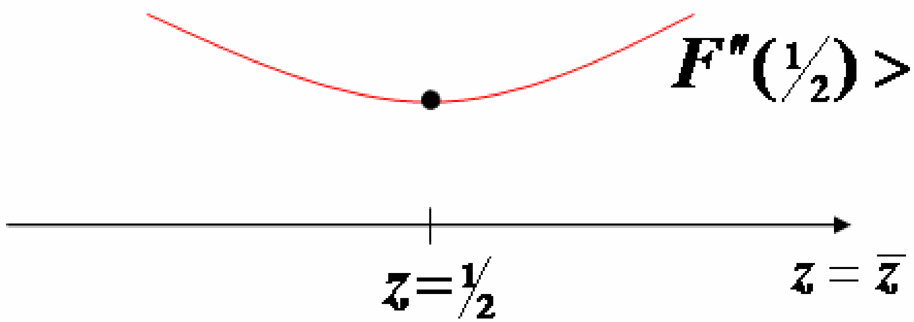

$F''(\frac{1}{2})>0$ for all

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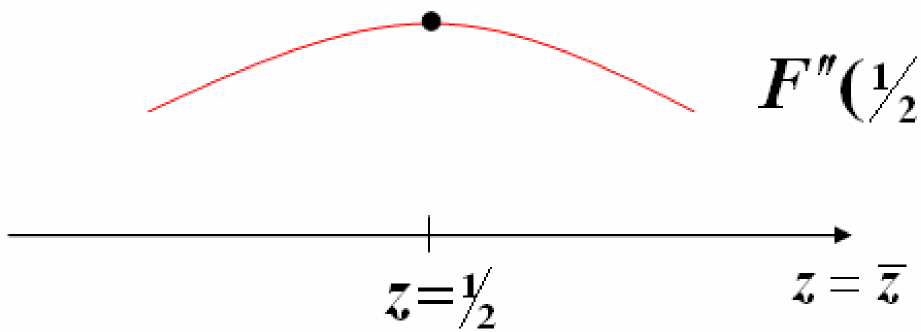
$\Rightarrow$ sum rule $1=\sum c^2_{\Delta,l} F_{d,\Delta,l}$ has no solutions

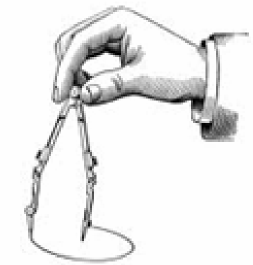


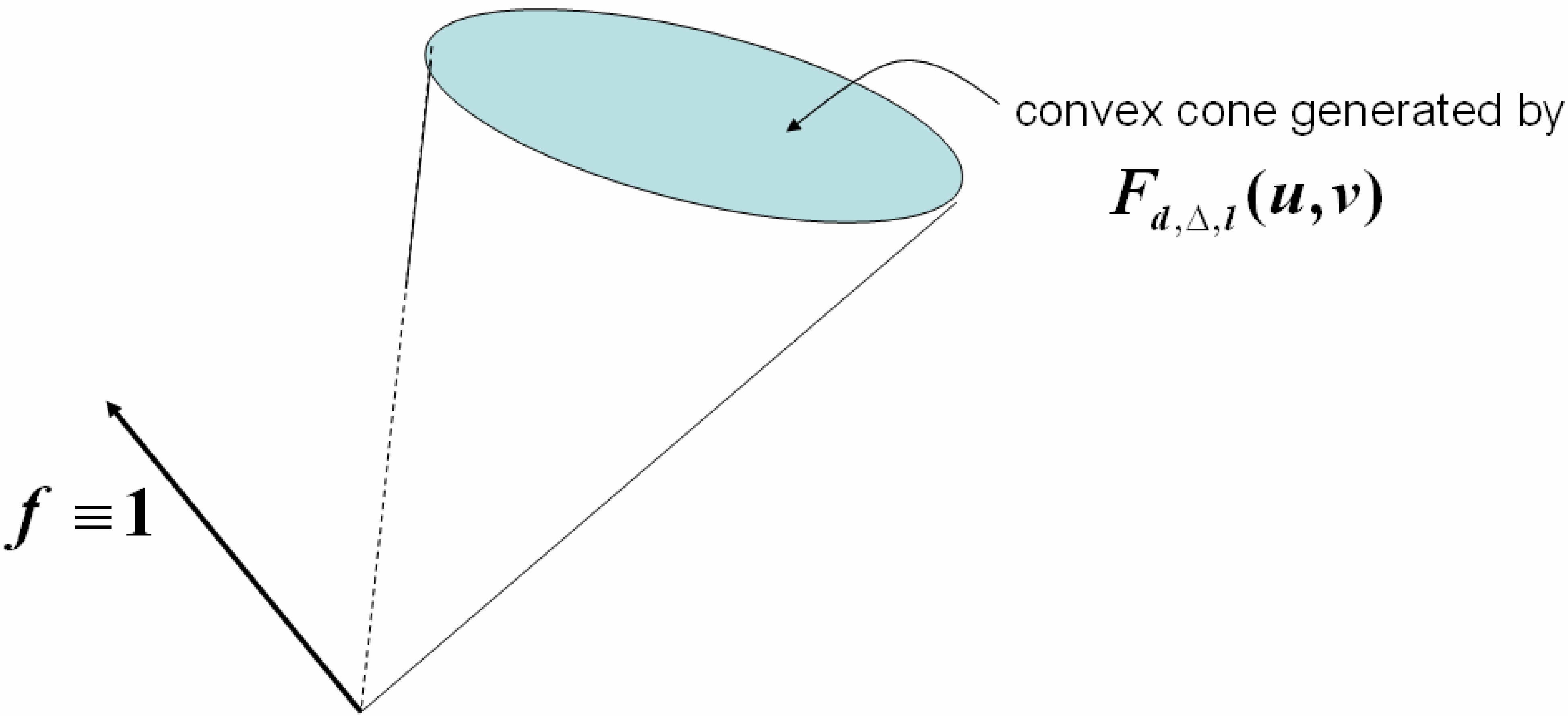
$F''(\tfrac{1}{2}) < 0$

$\Delta = 2,\ l = 0$

$\Rightarrow$ free scalar theory may exist

Sum rule: Geometric interpretation





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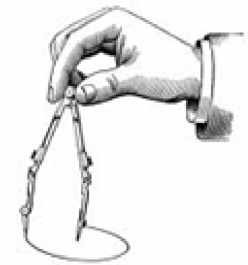

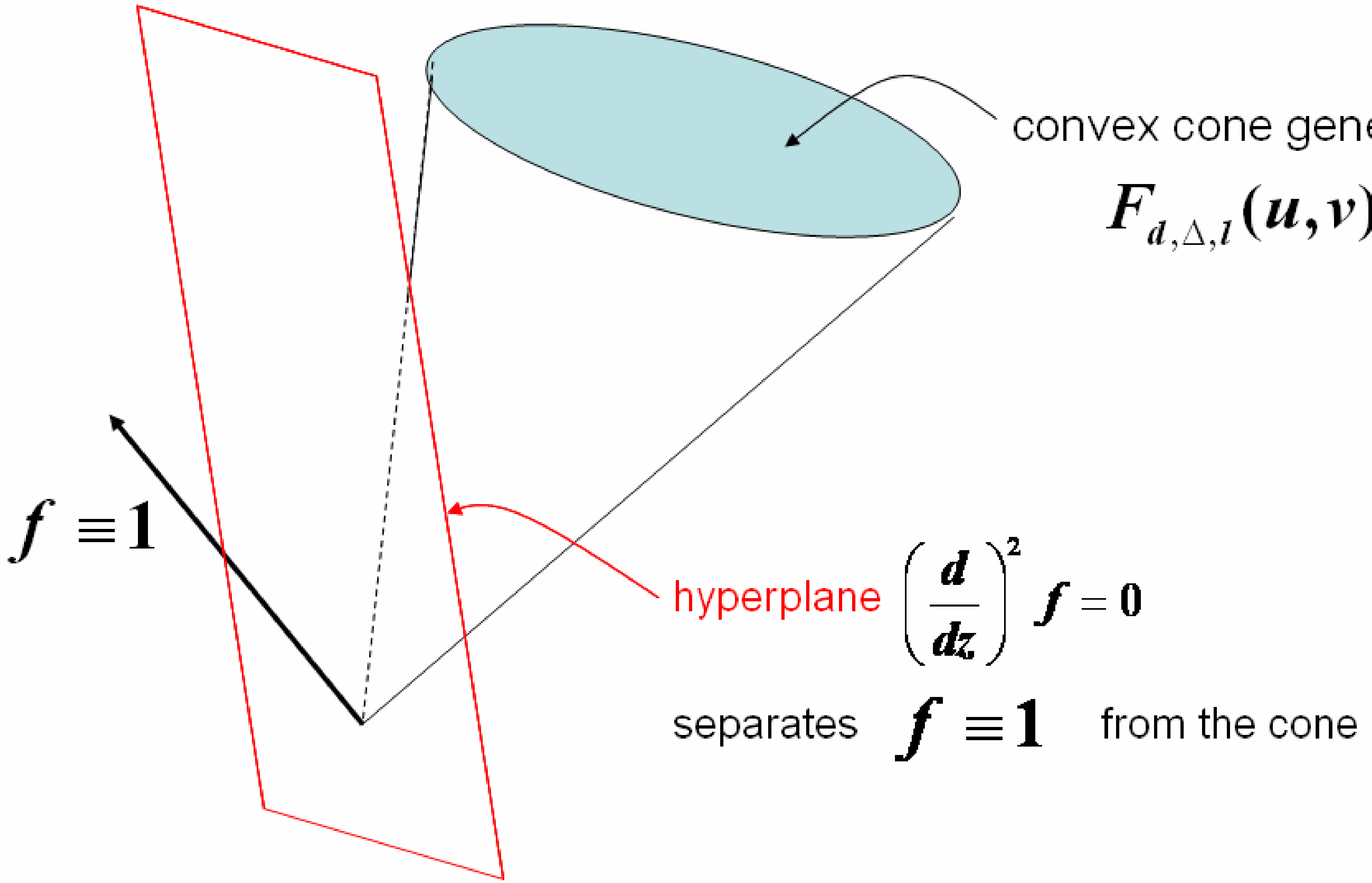

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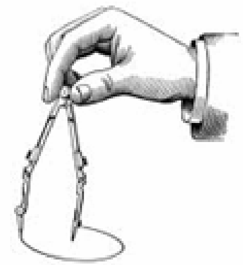

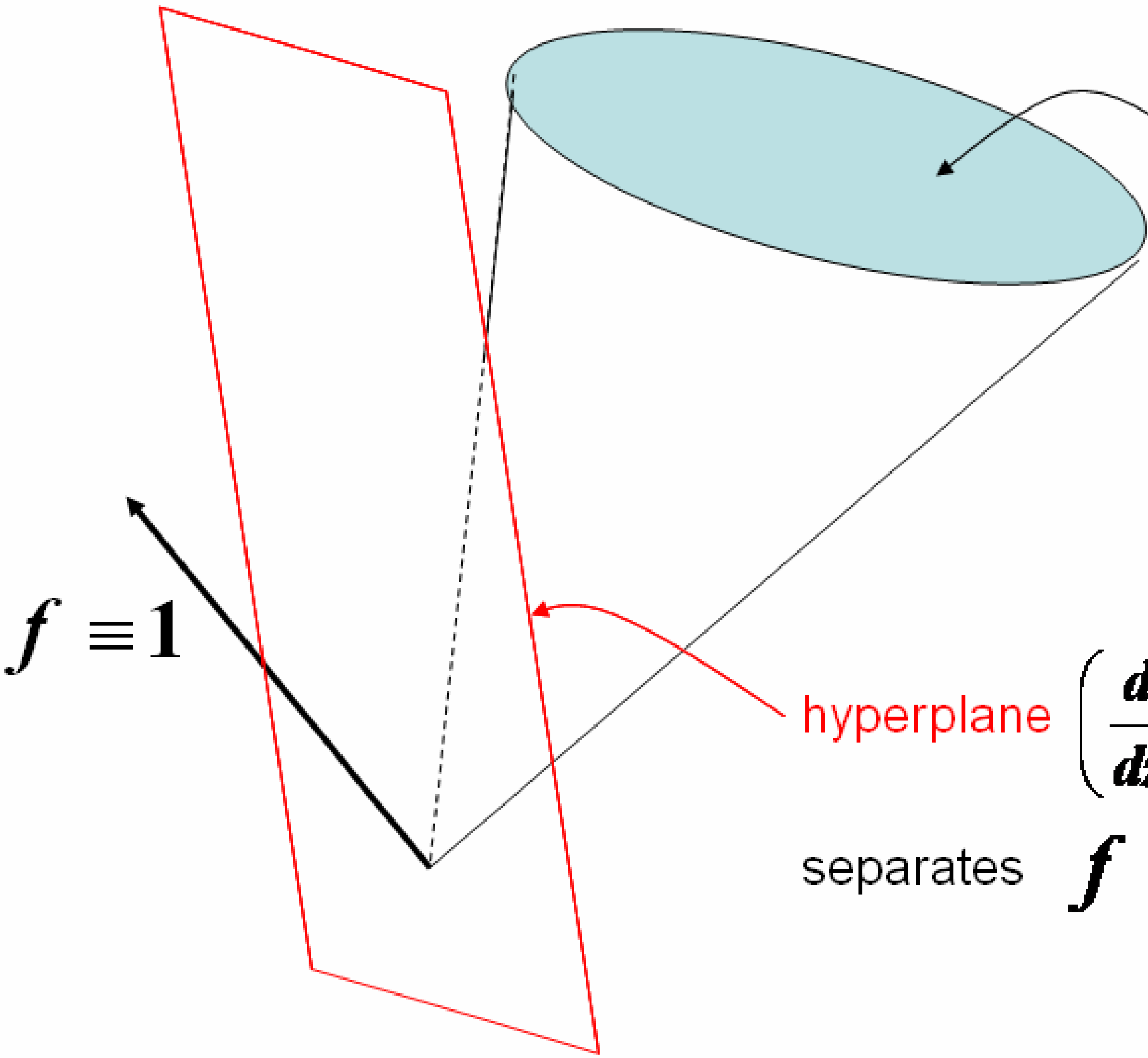

convex cone generated by $F_{d,\Delta,l}(u,v)$

hyperplane $\left(\frac{d}{dz}\right)^2 f = 0$

separates $f \equiv 1$ from the cone

More general linear combinations useful? $\sum \lambda_{m,n} \left(\frac{d}{dz}\right)^m \left(\frac{d}{d\bar{z}}\right)^n$

Generalization

Free field theory limit approached continuously:

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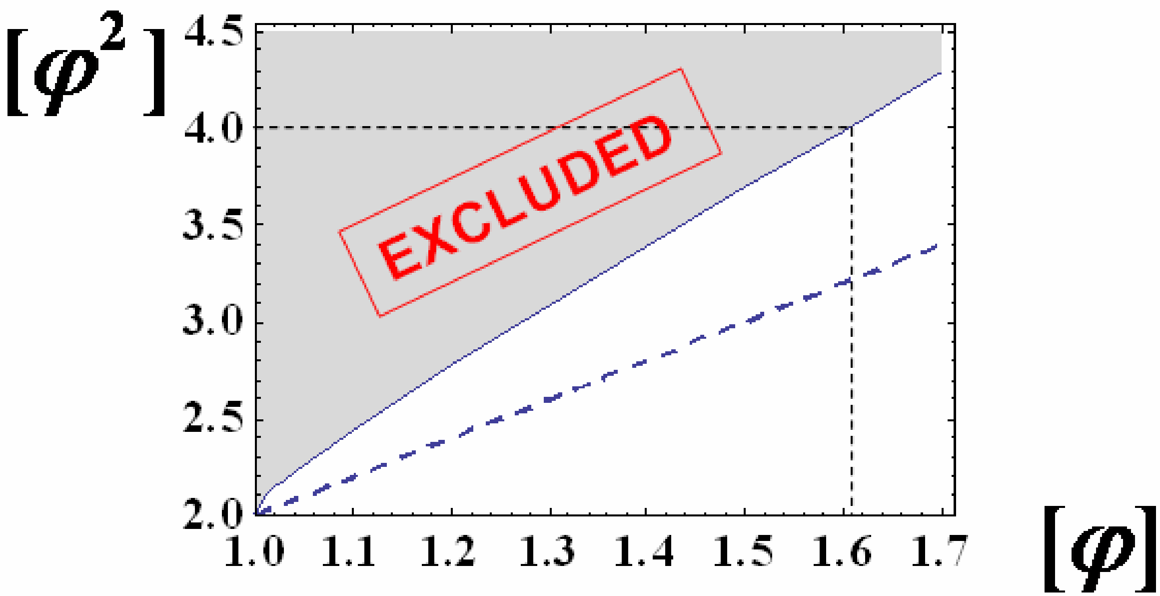

Another application

$$\phi_d \times \phi_d \supset c_{\phi\phi O} \cdot O_\Delta$$

$$\max | c_{\phi\phi O} | = f(d, \Delta) = ?$$

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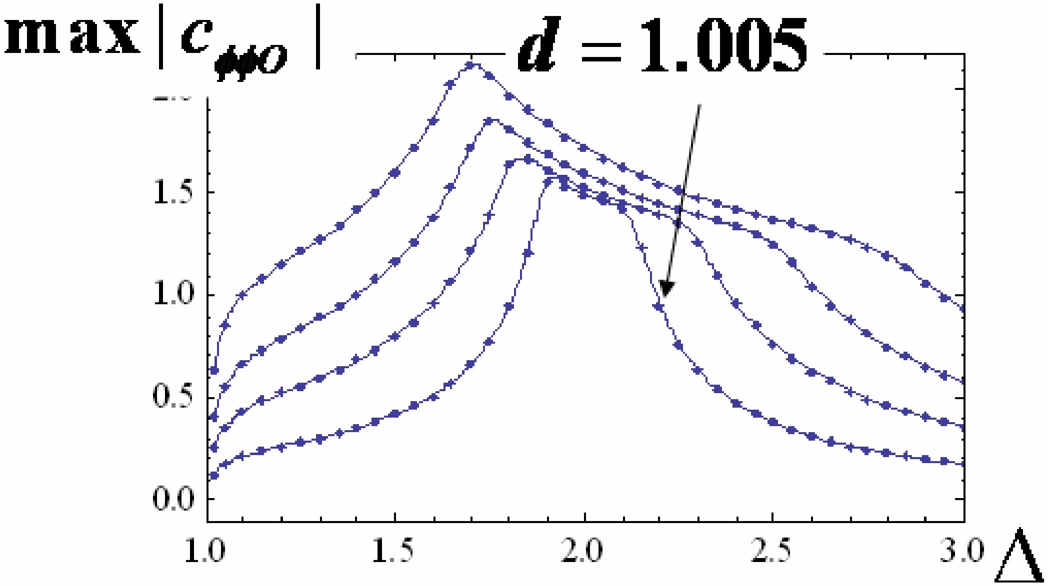

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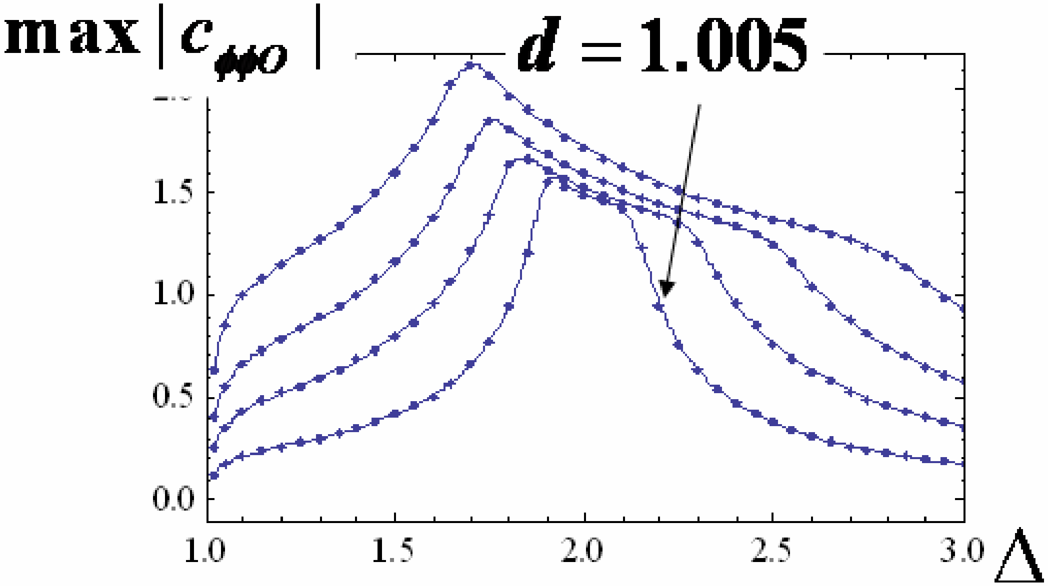

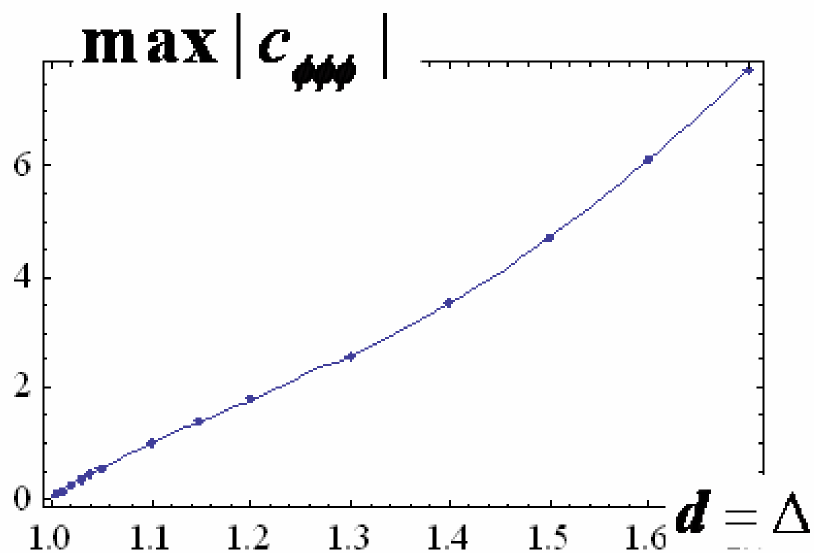

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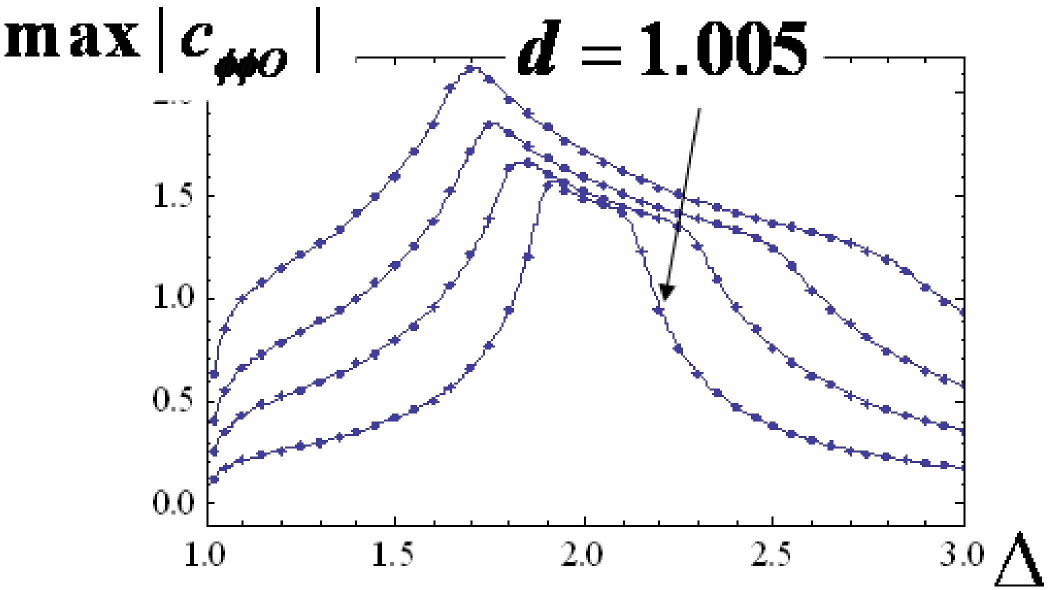

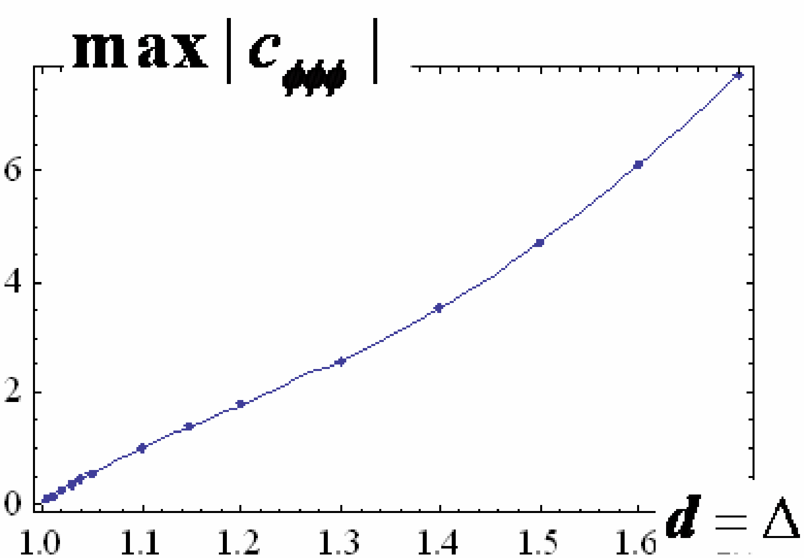

► 'Rigorous limits on the interaction strength in CFT'

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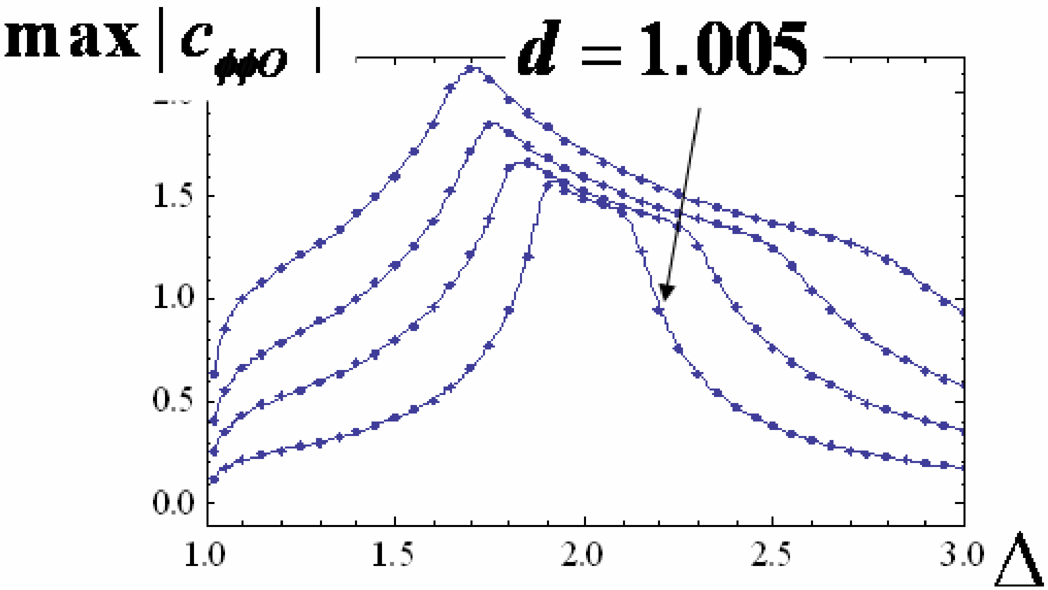
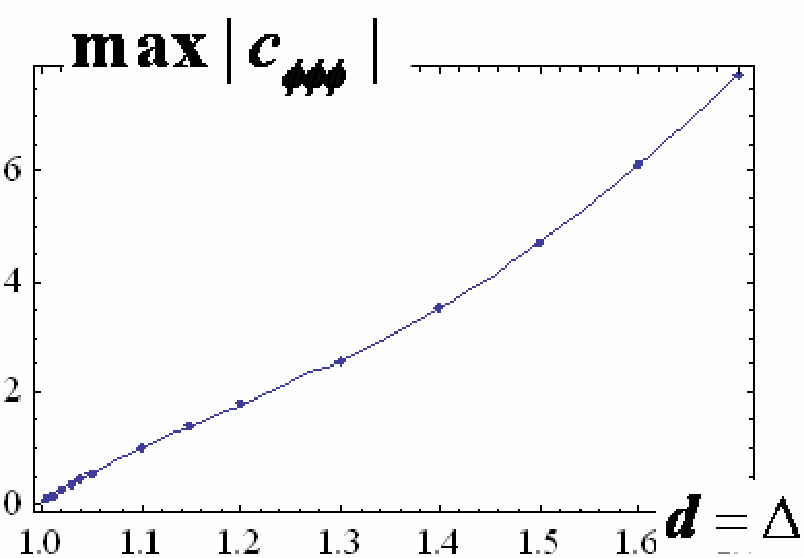
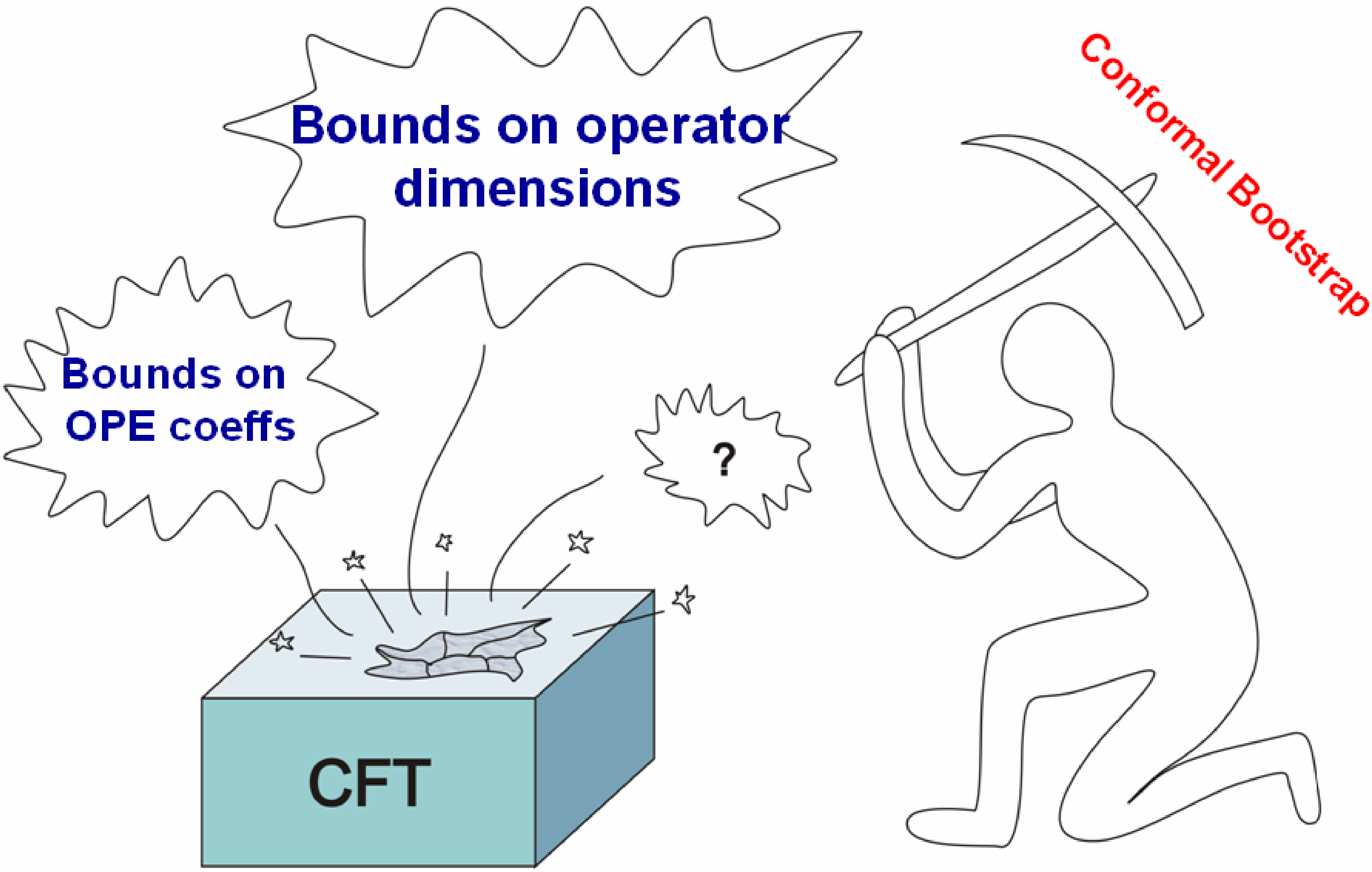




- ► 'Rigorous limits on the interaction strength in CFT'
- ► Important for unparticle phenomenology

Conclusions



BACKUP

2D and 3D examples

show that $\gamma_{\phi^2} >> \gamma_\phi$ is not impossible.

Ising model: $\sigma \times \sigma = 1 + \varepsilon$

2-dimensions (Onsager)	$[\sigma] = 1/8, \quad [\varepsilon] = 1$
3-dimensions (ϵ- and high-T expansions, Monte-Carlo)	$\gamma_\sigma \approx 0.02, \quad \gamma_\varepsilon \approx 0.4$

Extending analysis to 3d?

difficulty: finding 3d conformal blocks

(in odd dim's conformal blocks do not factorize as f(z)f(zbar))

Non-trivial extension for globally-symmetric case?

$$\phi_a \times \phi_b = \delta_{ab}(1 + O^{(1)}) + O^{(2)}{}_{ab} + \ldots$$

$$\ldots \supset J^{\mu}{}_{ab}$$

-two inequivalent crossing-symmetric 4-pt functions:

$$\langle \phi_1\phi_1\phi_1\phi_1 \rangle \quad \langle \phi_1\phi_2\phi_1\phi_2 \rangle$$

-OPE contains singlets and symmetric-traceless tensors (*even spin*); antisymmetric tensors (*odd spin*)

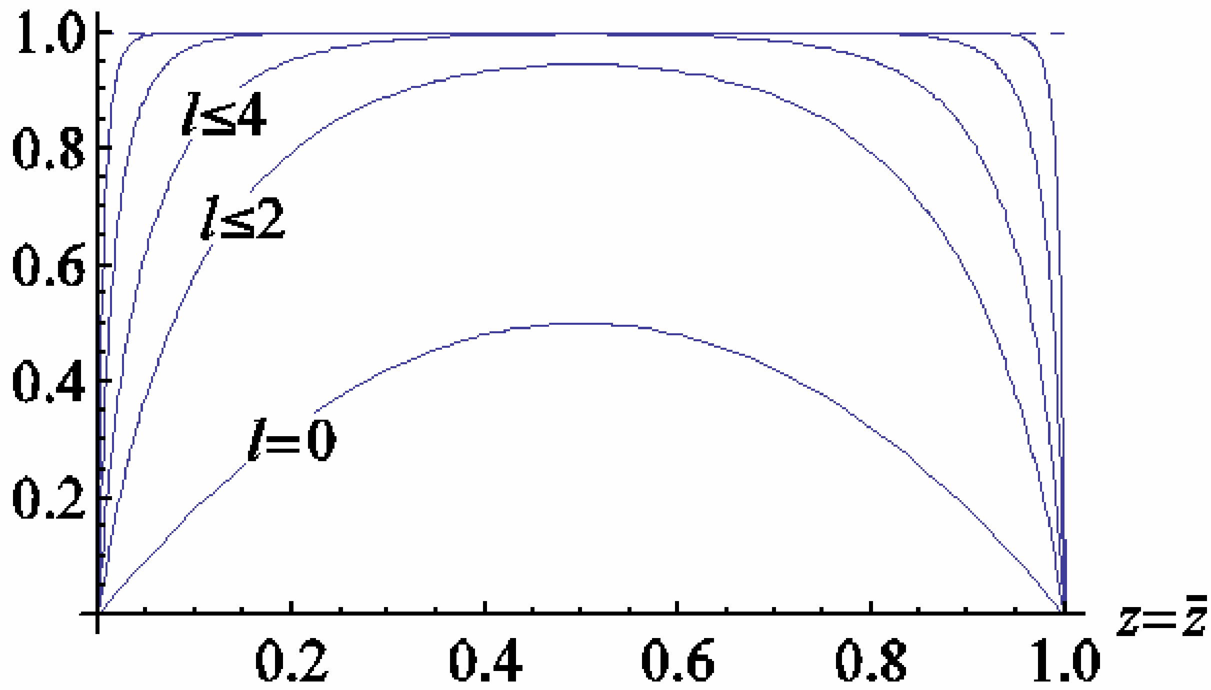
Can one bound $[O^{(1)}]$ in a model-independent way?

Sum rule convergence in free scalar theory

$$\phi \times \phi = \sum_{l=2n} \phi \overleftrightarrow{\partial}^{2n} \phi$$

twist 2 fields only

$$\lambda_l^2 = 2^{l+1} \frac{(l!)^2}{(2l!)^2}$$



Monotonic convergence