Three loop heavy quark potential

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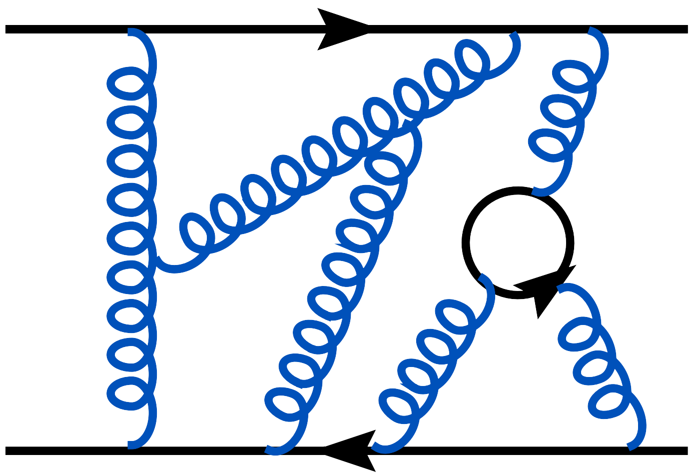
Interaction of 2 heavy quarks



Energy scales:

m	hard
$m\,v$	soft
$m\,v^2$	ultra-soft

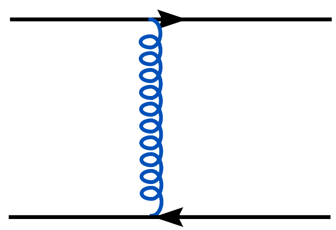
Disentangle scales ➪ effective theory

QCD $\rightarrow$ NRQCD $\rightarrow$ pNRQCD

[Caswell,Lepage'86] [Bodwin,Braaten,Lepage'95] [Pineda,Soto'98;Brambilla,Pineda,Soto,Vairo'00]

Static potential

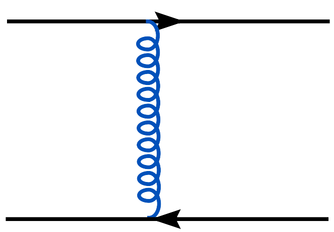
$$V(|\vec{q}|) \quad = \quad -\frac{4\pi C_F \alpha_s(|\vec{q}|)}{|\vec{q}|^2}\Bigg[1$$

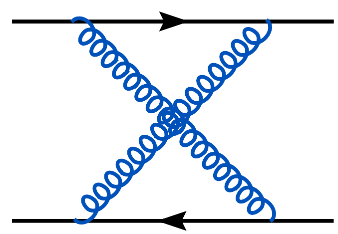


[Appelquist,Politzer'75,Susskind'77]

Static potential

$$V(|\vec{q}\,|) \;=\; -\frac{4\pi C_F \alpha_s(|\vec{q}\,|)}{|\vec{q}\,|^2}\Bigg[1 + \frac{\alpha_s(|\vec{q}\,|)}{4\pi} a_1$$

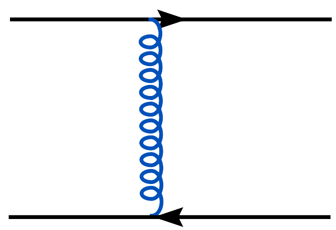


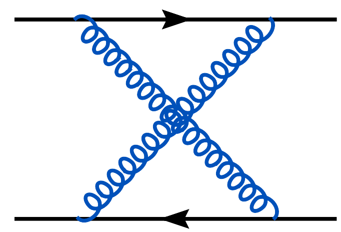


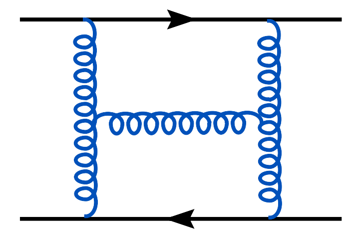
[Appelquist,Politzer'75,Susskind'77] [Fischler'77;Biloire'80]

Static potential

$$V(|\vec{q}|) \quad = \quad -\frac{4\pi C_F \alpha_s(|\vec{q}|)}{|\vec{q}|^2}\left[1 + \frac{\alpha_s(|\vec{q}|)}{4\pi}a_1 + \left(\frac{\alpha_s(|\vec{q}|)}{4\pi}\right)^2 a_2\right.$$



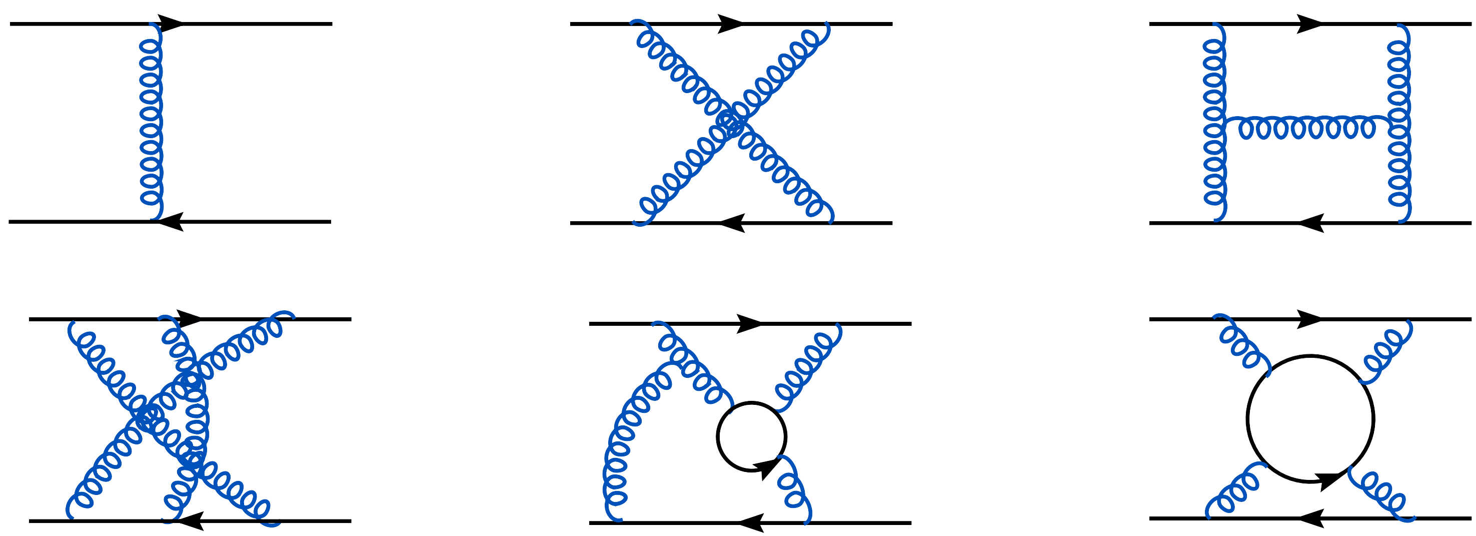




[Appelquist,Politzer’75,Susskind’77] [Fischler’77;Biloire’80] [Peter’96;Schröder’98]

Static potential

$$V(|\vec{q}\,|) \;=\; -\frac{4\pi C_F \alpha_s(|\vec{q}\,|)}{|\vec{q}\,|^2}\Bigg[1 + \frac{\alpha_s(|\vec{q}\,|)}{4\pi} a_1 + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^2 a_2 + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^3 \left(a_3 + 8\pi^2 C_A^3 \ln\frac{\mu^2}{\vec{q}^{\,2}}\right) + \cdots\Bigg]$$



[Appelquist,Politzer'75,Susskind'77] [Fischler'77;Biloire'80] [Peter'96;Schröder'98] [Smirnov,Smirnov,Steinhauser'08] [Smirnov,Smirnov,Steinhauser'09] [Anzai,Kiyo,Sumino'09]

Static potential

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- IR divergence [Appelquist,Dine,Muzinich'77]
  (in "naive perturbation theory")
- pNRQCD: ultra-soft contribution:
  (us)-gluons and $(Q\bar{Q})$ bound states as dynamical degrees of freedom ➪ IR finite result

  [Brambilla,Pineda,Soto,Vairo'99; Kniehl,Penin,Smirnov,Steinhauser'02]
- Finite physical quantities (e.g.: $E_{q\bar{q}}$)
- H.O. log-contributions to V: [Pineda,Soto'00; Brambilla,Garcia i Tormo,Soto,Vairo'07]

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a_3 needed for ...

- $E_{q\bar{q}}$ [NNNLO: Penin,Steinhauser'02; Kiyo,Sumino'03]
  - $M_{\Upsilon(1S)} = 2m_b + E_{b\bar{b}}^{n=1}$ ⇨ m_b
  - $E_{\rm res} = 2m_t + E_{t\bar{t}}^{n=1} + \delta^{\Gamma_t} E_{\rm res}$ ⇨ m_t
- $\sigma(e^+e^- \to t\bar{t})$, NNNLO

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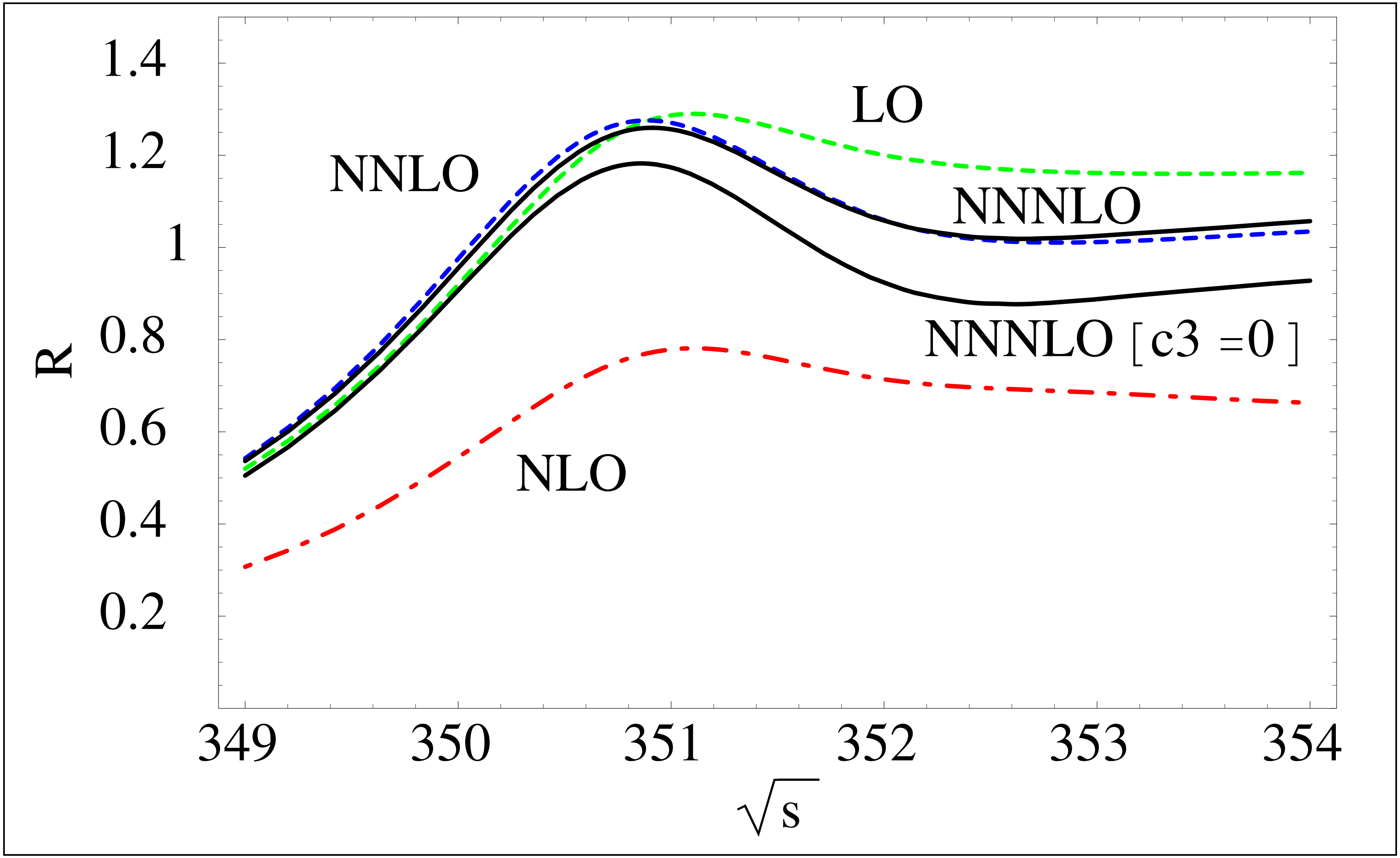
SFB TR9

a_3 needed for ...



[Beneke,Kiyo,Schuller'08]

a_3 needed for ...

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- $\sigma(e^+e^- \to t\bar{t})$, NNNLO
- Comparison of pQCD and LQCD

  [Necco,Sommer'01,Pineda'02,Sumino'05, . . . , Brambilla,Vairo,Garcia i Tormo,Soto'09]

a_3 needed for . . .

- $E_{q\bar{q}}$ [NNNLO: Penin,Steinhauser'02; Kiyo,Sumino'03]
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  [Necco,Sommer'01,Pineda'02,Sumino'05, . . . , Brambilla,Vairo,Garcia i Tormo,Soto'09]
- Fundamental quantity

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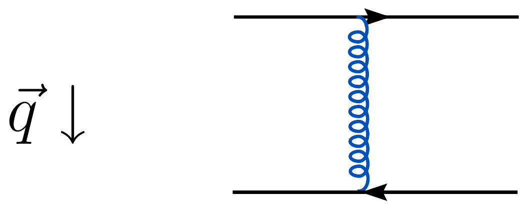
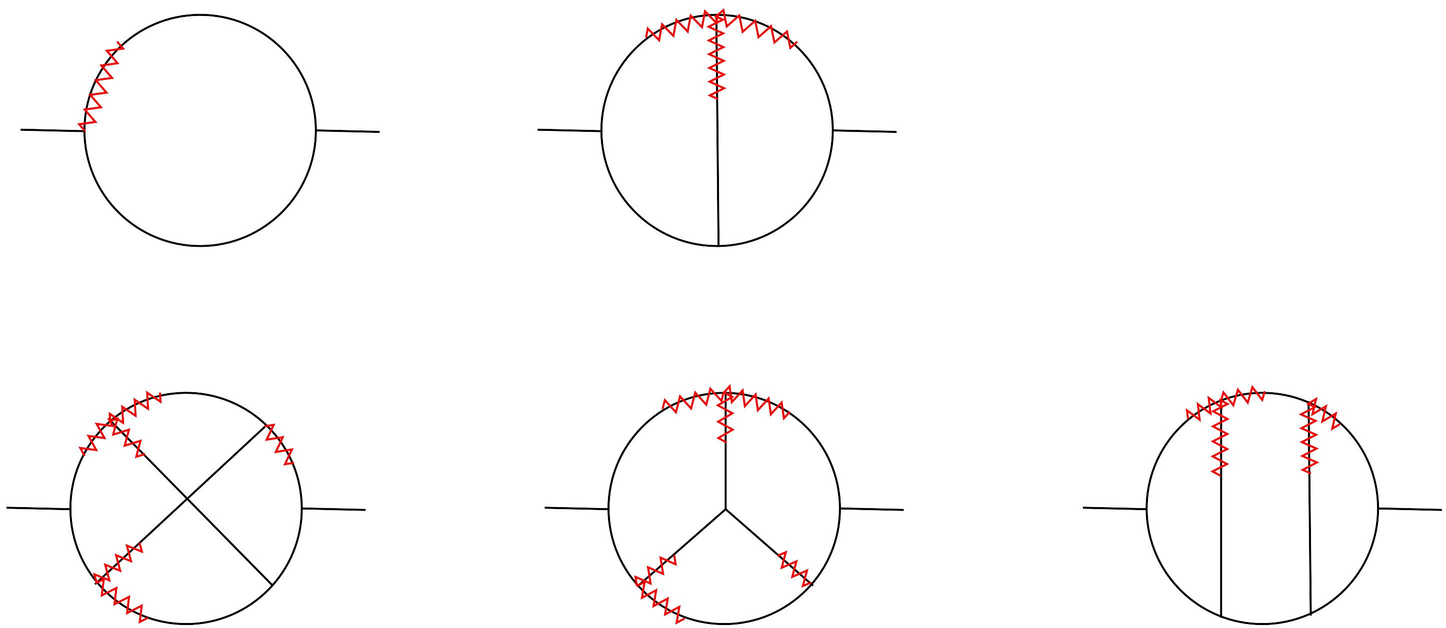
Loop integrals



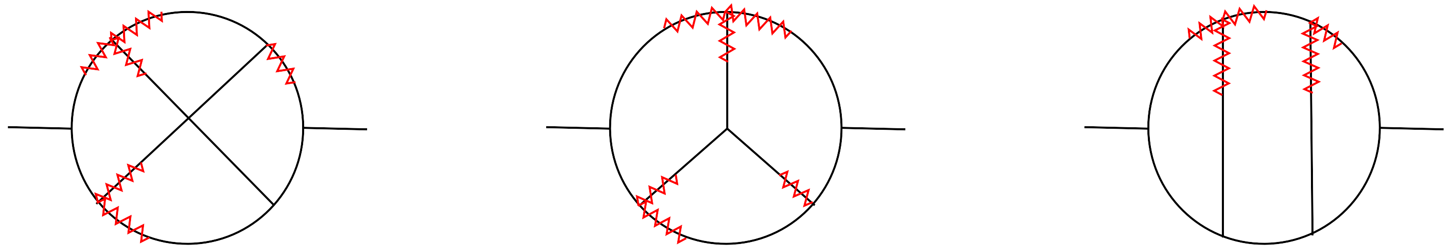
1 scale:
momentum transfer between
quark and anti-quark



➪ up to 14 lines + numerator ➪ 15 indices for reduction

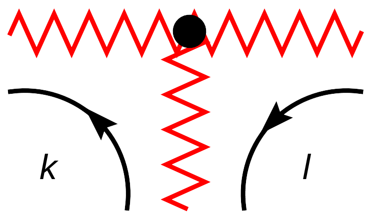
Loop integrals (2)

3 loops:



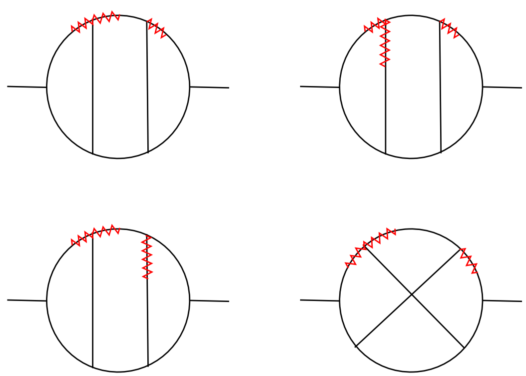
➪ (up to) 8+6=14 lines ➪ 14+1=15 indices (reduction)

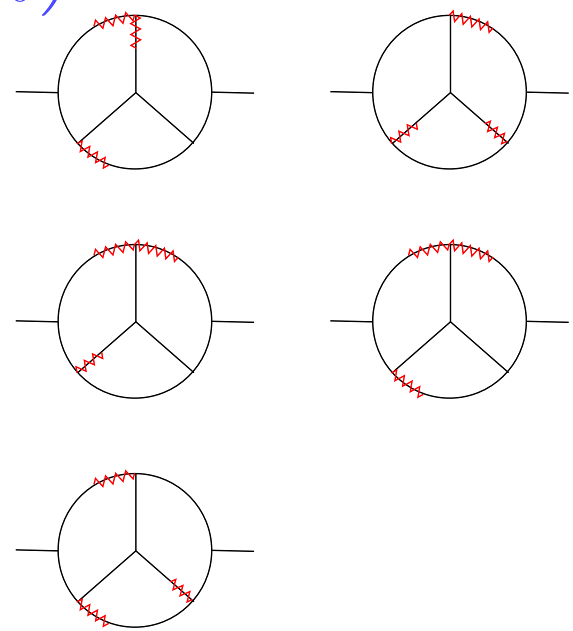
trick:



$$\frac{1}{k_0 l_0} = \frac{1}{k_0 - l_0}\left(\frac{1}{l_0} - \frac{1}{k_0}\right)$$

➪ at most 12 indices





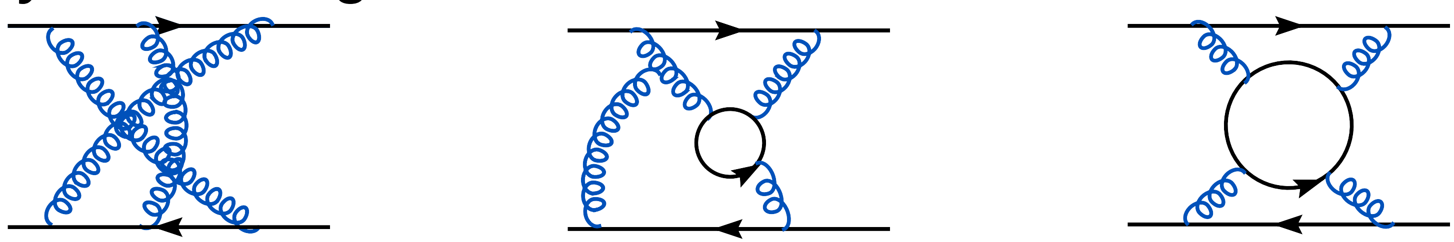
But: more cases to be considered . . .

The methods

- 1 loop: $\surd$
- 2 loops:

  [Schröder'98]: IBP $\to$ MIs

  [Smirnov,Steinhauser'03]: "Baikov" method
- 3 loops:
  - 8100 Feynman diagrams

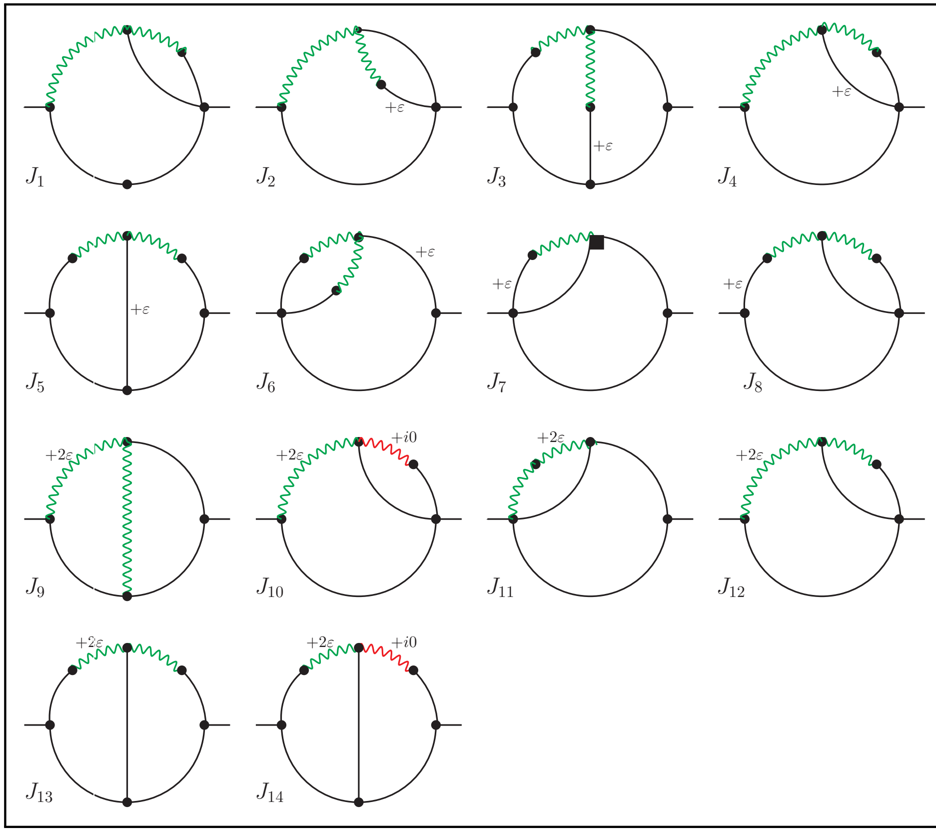


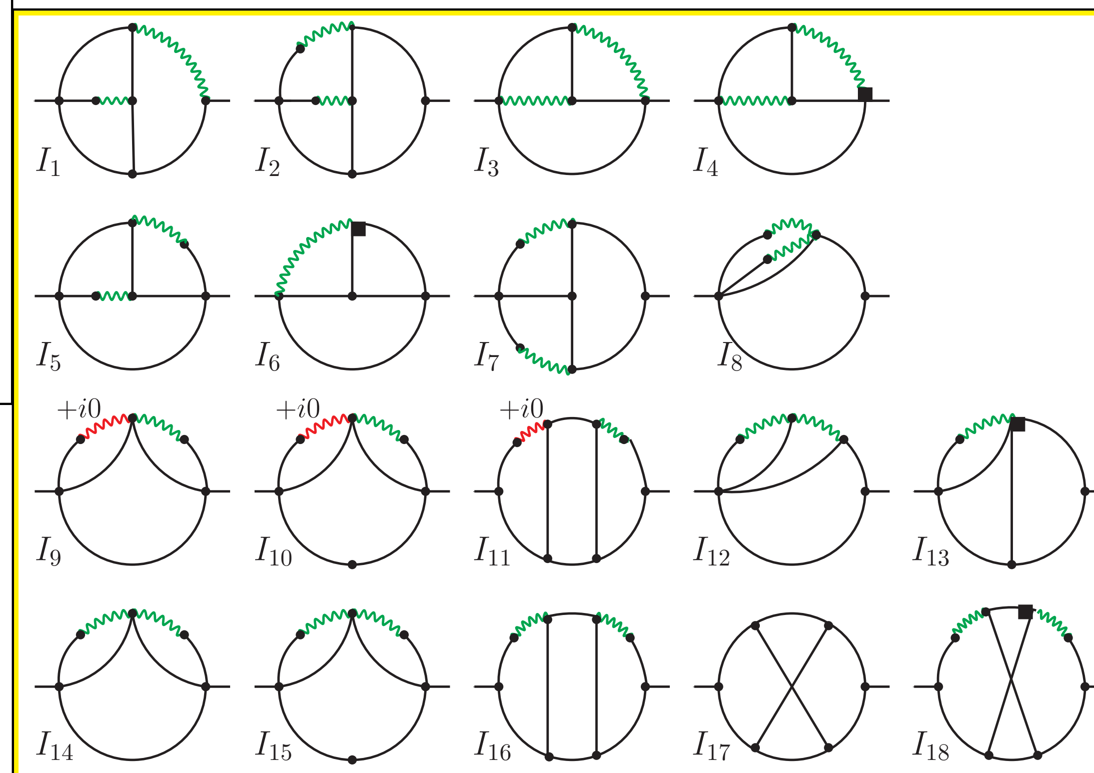
  - qgraf, q2e/exp, FORM

    [Nogueira'91],[Harlander,Seidelsticker,Steinhauser'97'99], [Vermaseren]
  - FIRE [Smirnov'08]: Gröbner $\oplus$ Laporta

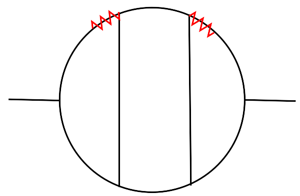
    Reduction to master integrals

Master integrals

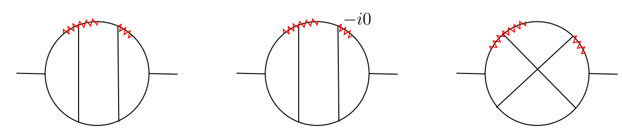




Master integrals (2)

- Mellin Barnes (see also: MB.m [Czakon'06])
- `PSLQ` [Ferguson,Bailey'91]
- example:  ➪ 6-fold MB representation

$$➪ \; -\frac{(i\pi^{d/2})^3}{(-q^2)^{3+3\epsilon}v^2}\left[\frac{56\pi^4}{135\epsilon}+\frac{112\pi^4}{135}+\frac{16\pi^2\zeta(3)}{9}+\frac{8\zeta(5)}{3}+O(\epsilon)\right]$$

- all but 3 coefficients analytically 

- all MIs checked with `FIESTA` [Smirnov,Tentyukov'08; Smirnov,Smirnov,Tentyukov'09]

  example: "analytic" :: "analytic"–"FIESTA"

```
ep^-1 :: -40.4067 :: 0.00010 +- 0.00044
ep^0  :: -104.67  :: 0.01052 +- 0.00591
```

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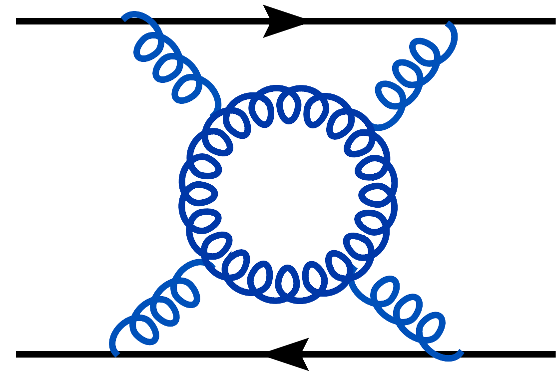
ξ dependence



Individual diagrams: up to ξ^6

a_3: gauge parameter independent

	ξ^1	—	ξ^2	—	ξ^3	—	ξ^4	—	ξ^5	—	ξ^6
n_l	$\surd$		$\surd$		$\surd$		$\surd$		$\surd$		$\surd$
C_A^3	$\surd$		$\surd$		$\surd$		$\surd$		$\surd$		$\surd$
$d_F^{abcd} d_A^{abcd}$	$\surd$		$\surd$		$\surd$		$\surd$		$\surd$		$\surd$

Complexity: number of occurring integrals (normalized to ξ^0 case)

ξ^0	ξ^1	ξ^2	ξ^3	ξ^4	ξ^5	ξ^6
1	7	16	18	11	3	0.2

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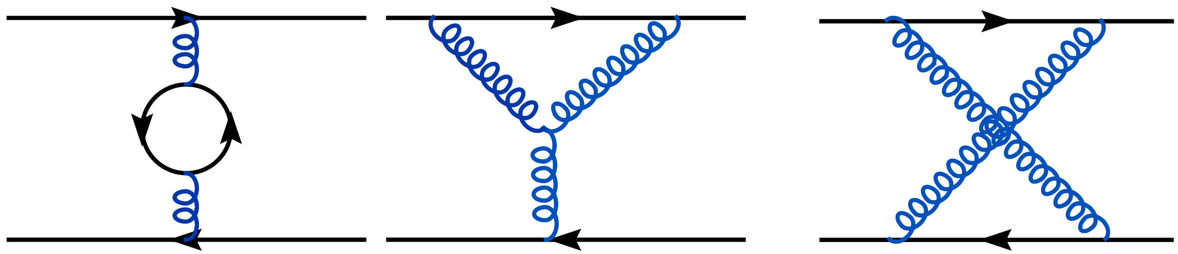
Structure of a_1, a_2, a_3

$$a_1 \quad = \quad \frac{31}{9} C_A - \frac{20}{9} T_F n_l$$



Abelian part from iterations of lower orders

$$a_2 : C_A^2, C_A T_F n_l, C_F T_F n_l, (T_F n_l)^2$$

$$a_3 : C_A^3, \frac{d_F^{abcd} d_A^{abcd}}{N_A}, \qquad C_A^2 T_F n_l, C_A C_F T_F n_l, C_F^2 T_F n_l, \frac{d_F^{abcd} d_F^{abcd}}{N_A}, \\ C_A (T_F n_l)^2, C_F (T_F n_l)^2, (T_F n_l)^3$$

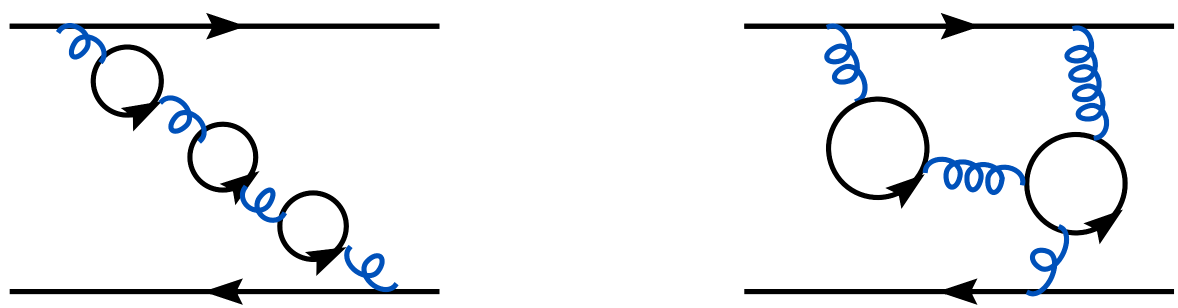
Results

$$a_3 = a_3^{(3)} n_l^3 + a_3^{(2)} n_l^2 + a_3^{(1)} n_l + a_3^{(0)}$$

[Smirnov,Smirnov,Steinhauser'08; Smirnov,Smirnov,Steinhauser'09]

$$a_3^{(3)} = -\left(\frac{20}{9}\right)^3 T_F^3$$

$$a_3^{(2)} = \left(\frac{12541}{243} + \frac{368\zeta(3)}{3} + \frac{64\pi^4}{135}\right) C_A T_F^2 + \left(\frac{14002}{81} - \frac{416\zeta(3)}{3}\right) C_F T_F^2$$



$$C_A = N_c\,, \quad C_F = \frac{N_c^2 - 1}{2N_c}\,, \quad T_F = \frac{1}{2}\,, \quad \frac{d_F^{abcd} d_F^{abcd}}{N_A} = \frac{18 - 6N_c^2 + N_c^4}{96 N_c^2}$$

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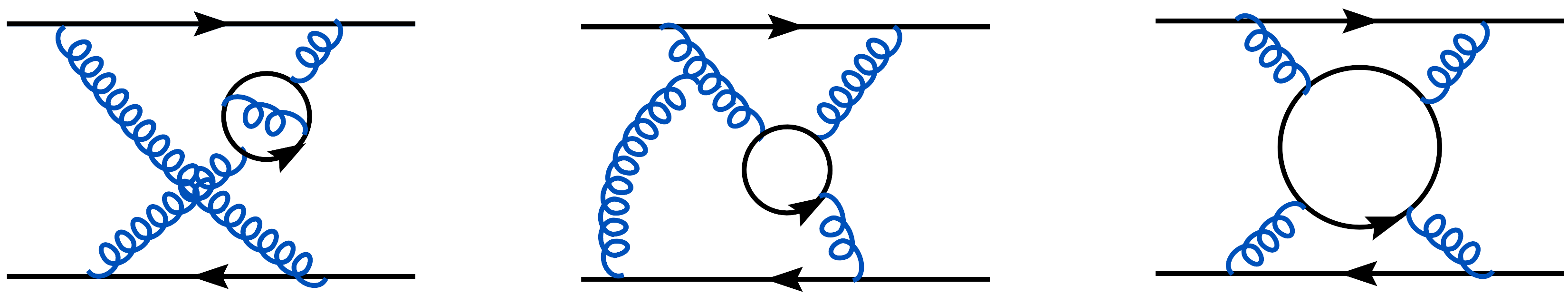
$$a_3^{(1)} = -709.717 C_A^2 T_F + \left(-\frac{71281}{162} + 264\zeta(3) + 80\zeta(5)\right) C_A C_F T_F + \left(\frac{286}{9} + \frac{296\zeta(3)}{3} - 160\zeta(5)\right) C_F^2 T_F - 56.83(1)\frac{d_F^{abcd} d_F^{abcd}}{N_A}$$



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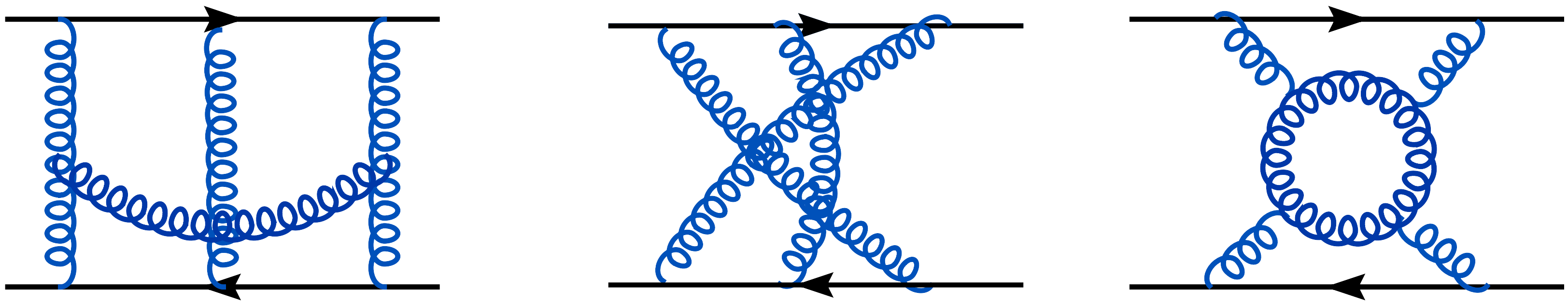
($a_3^{(0)}$: independet calculation: [Anzai,Kiyo,Sumino'09])

$$a_3^{(0)} = 502.24(1) C_A^3 - 136.39(6) \frac{d_F^{abcd} d_A^{abcd}}{N_A}$$



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$$\frac{d_F^{abcd} d_A^{abcd}}{N_A} = \frac{N_c^3 + 6N_c}{48}$$

Numerical results

$$V(|\vec{q}|) = -\frac{4\pi C_F \alpha_s(|\vec{q}|)}{\vec{q}^{\,2}} \Bigg[1 + \frac{\alpha_s}{\pi}(2.5833 - 0.2778 n_l) + \left(\frac{\alpha_s}{\pi}\right)^2 (28.5468 - 4.1471 n_l + 0.0772 n_l^2) + \left(\frac{\alpha_s}{\pi}\right)^3 (209.884(1) - 51.4048 n_l + 2.9061 n_l^2 - 0.0214 n_l^3) + \cdots \Bigg]$$

	n_l	$\alpha_s^{(n_l)}$	1 loop	2 loop	3 loop
charm	3	0.40	0.2228	0.2723	0.1677
bottom	4	0.25	0.1172	0.08354	0.02489
top	5	0.15	0.05703	0.02220	0.002485

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$E_{q\bar{q}}$

$$E_1^{\rm p.t.} = E_1^C + \delta E_1^{(1)} + \delta E_1^{(2)} + \delta E_1^{(3)} + \ldots\,,$$

$$E_1^C = -\frac{C_F^2\alpha_s^2 m_q}{4}$$

$$\mu_S = C_F\alpha_s(\mu_S)m_q$$

$$\delta E_1^{(3)}\Big|_{\rm charm} = \alpha_s^3 E_1^C\left(129.79 + 5.241\Big|_{a_3} + 15.297\ln(\alpha_s)\right)$$

$$\delta E_1^{(3)}\Big|_{\rm bottom} = \alpha_s^3 E_1^C\left(104.82 + 3.186\Big|_{a_3} + 15.297\ln(\alpha_s)\right)$$

$$\delta E_1^{(3)}\Big|_{\rm top} = \alpha_s^3 E_1^C\left(83.386 + 1.473\Big|_{a_3} + 15.297\ln(\alpha_s)\right)$$

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Comparison with approximations

$$V(|\vec{q}|) = -\frac{4\pi C_F \alpha_s(|\vec{q}|)}{\vec{q}^{\,2}} \Big[1 + \ldots$$
$$+ \left(\frac{\alpha_s}{\pi}\right)^3 \left(209.884(1) - 51.4048 n_l + 2.9061 n_l^2 - 0.0214 n_l^3\right)$$
$$+ \cdots \Big]$$

$n_l = 0$	1	2	3	4	5	6		
210	161	119	81.2	49.4	22.8	1.4	exact	[Smirnov et al.'09]
313	250	193	142	97.5	60.1	30.5	Padé appr.	[Chishtie,Elias'01]
292	227	168	116	72	37	12	renormalon	[Pineda'01]

fit to cubic polynomial in n_l ⇨

$$a_3/4^3 \approx 380.9 - 70.42\, n_l + 2.34\, n_l^2 + 0.08\, n_l^3$$
$$a_3/4^3 \approx 362.0 - 72.17\, n_l + 2.00\, n_l^2 + 0.17\, n_l^3$$

Comparison with approximations

$$V(|\vec{q}|) = -\frac{4\pi C_F \alpha_s(|\vec{q}|)}{\vec{q}^{\,2}} \Big[1 + \dots + \left(\frac{\alpha_s}{\pi}\right)^3 \left(209.884(1) - 51.4048 n_l + 2.9061 n_l^2 - 0.0214 n_l^3\right) + \cdots \Big]$$

Comparison: pQCD $\leftrightarrow$ lattice (for static energy) [Necco,Sommer'01]

➪ extract $a_3^{(0)}$

Coordinate space: $V(r) = -C_F \frac{\alpha_s}{r} \Big[1 + \dots + \tilde{a}_3 + \dots \Big]$

➪ $\tilde{a}_3 = 1.11^{+0.06}_{-0.03} \cdot 10^5$ [Brambilla,Vairo,Garcia i Tormo,Soto'09]

Momentum space ➪ $a_3^{(0)}/4^3 = 202 \dots 337$

[$\leftrightarrow$ "209.884(1)"]

Conclusions

- 3-loop static potential computed (a_3) in pQCD
- Building block for various "quarkonium observables"
  $E_{q\bar{q}}$, $\sigma(e^+e^- \to t\bar{t})$, ...
- Fundamental quantity of QCD

- Aim: completely analytical result