Quark and Lepton Evolution Invariants in the Standard Model

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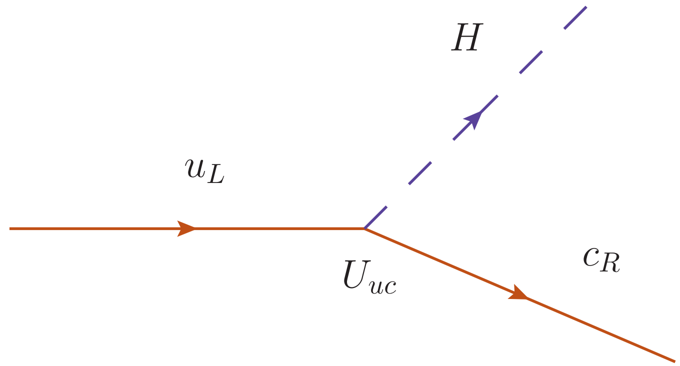
* With Rama Krishnan and Bill Scott, arXiv:1007.3810 [hep-ph]

Outline of Talk

- Introduction - SM Yukawa Couplings
- Evolution of Yukawa Couplings
- RG Evolution Invariants
- Exact One-Loop Evolution Invariants in the SM
- Summary and Conclusions

Introduction - SM Yukawa Couplings

- SM has 9 complex Yukawa couplings between Higgs and charge 2/3 quarks - written as a 3×3 matrix, U.
- 9 more for charge -1/3 quarks, D.
- Together determine the quark masses and V_{CKM} mixing angles



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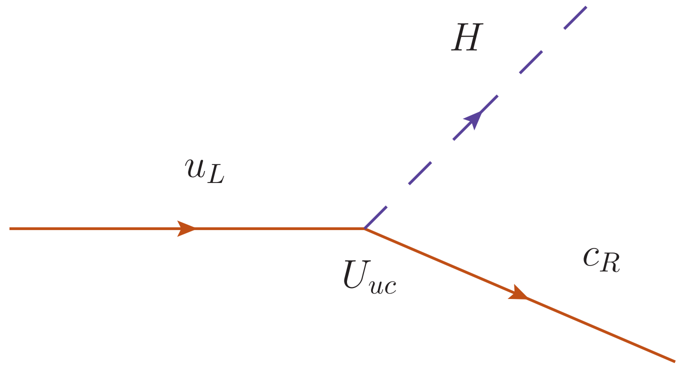

- Useful to define also their Hermitian-squares:

$$\mathcal{U} = U^\dagger U; \qquad \mathcal{D} = D^\dagger D$$

- With Dirac νs, story is similar, with Hermitian-squared mass matrices, $\mathcal{L}$ and $\mathcal{N}$ for charged leptons and νs respectively.

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$$U^{-1}\frac{dU}{dt} = \gamma_u + \frac{3}{2}(\mathcal{U} - \mathcal{D}); \qquad D^{-1}\frac{dD}{dt} = \gamma_d + \frac{3}{2}(\mathcal{D} - \mathcal{U})$$

where

$$\gamma_u = T - (\frac{17}{12}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2,); \qquad \gamma_d = T - (\frac{5}{12}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2),$$

with:

$$T = Tr(3\mathcal{U} + 3\mathcal{D} + \mathcal{N} + \mathcal{L}).$$

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- Control how the quark masses and mixing angles evolve

- Analogous equations for leptons in Dirac ν case

Evolution of Yukawa Couplings (cont.)

- Above RGEs for Yukawas are coupled and non-linear.
- Can be solved for observables at high (eg. GUT) scales
- But exptl. errors in poorly-known observables feed into solns. for others, rendering their high-energy values poorly-determined
- eg. m_t known to 1.7% at m_Z, but only to 5.4% at M_{GUT}

RG Evolution Invariants

Recent interest in *RG evolution invariants*, observables which do not evolve with energy.

Useful features:

- Exptly.-determined values (at eg. weak scale) are valid at all scales.
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We will derive EXACT evolution invariants in the SM (at one-loop order).

They will relate masses and mixings in a new way.

Eigenvalues of $\mathcal{UD}$

Eigenvalues, λ_i, of the product matrix $\mathcal{UD}$ are given by its eigenvalue equation:

$$\lambda_i^3 - T_{\mathcal{UD}} \cdot \lambda_i^2 + P_{\mathcal{UD}} \cdot \lambda_i - D_{\mathcal{UD}} = 0$$

where coefficients

$$T_{\mathcal{UD}} = Tr(\mathcal{UD}); \quad P_{\mathcal{UD}} = \frac{1}{2}(Tr^2(\mathcal{UD}) - Tr(\mathcal{UD})^2); \quad D_{\mathcal{UD}} = Det(\mathcal{UD})$$

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They have *pure* RGEs (unlike for $\mathcal{U}$ and $\mathcal{D}$ separately):

$$\frac{dT_{\mathcal{UD}}}{dt} = 2(\gamma_u + \gamma_d)T_{\mathcal{UD}}$$

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So, eigenvalues, λ_i, of $\mathcal{UD}$ have pure evolutions with coefficient $2(\gamma_u + \gamma_d)$ - new result.

Exact One Loop Evolution Invariants in the SM

The three $\mathcal{UD}$ coefficients have pure evolutions with rate proportional to their order.

May thus form two independent "dimensionless" combinations which are exact one-loop evolution invariants:

$$\mathcal{I}^q_{TD} \equiv \frac{T_{\mathcal{UD}}}{D_{\mathcal{UD}}^{\frac{1}{3}}}; \qquad \mathcal{I}^q_{PD} \equiv \frac{P_{\mathcal{UD}}}{D_{\mathcal{UD}}^{\frac{2}{3}}}; \qquad \text{with } \frac{d\mathcal{I}^q_{TD}}{dt} = \frac{d\mathcal{I}^q_{PD}}{dt} = 0.$$

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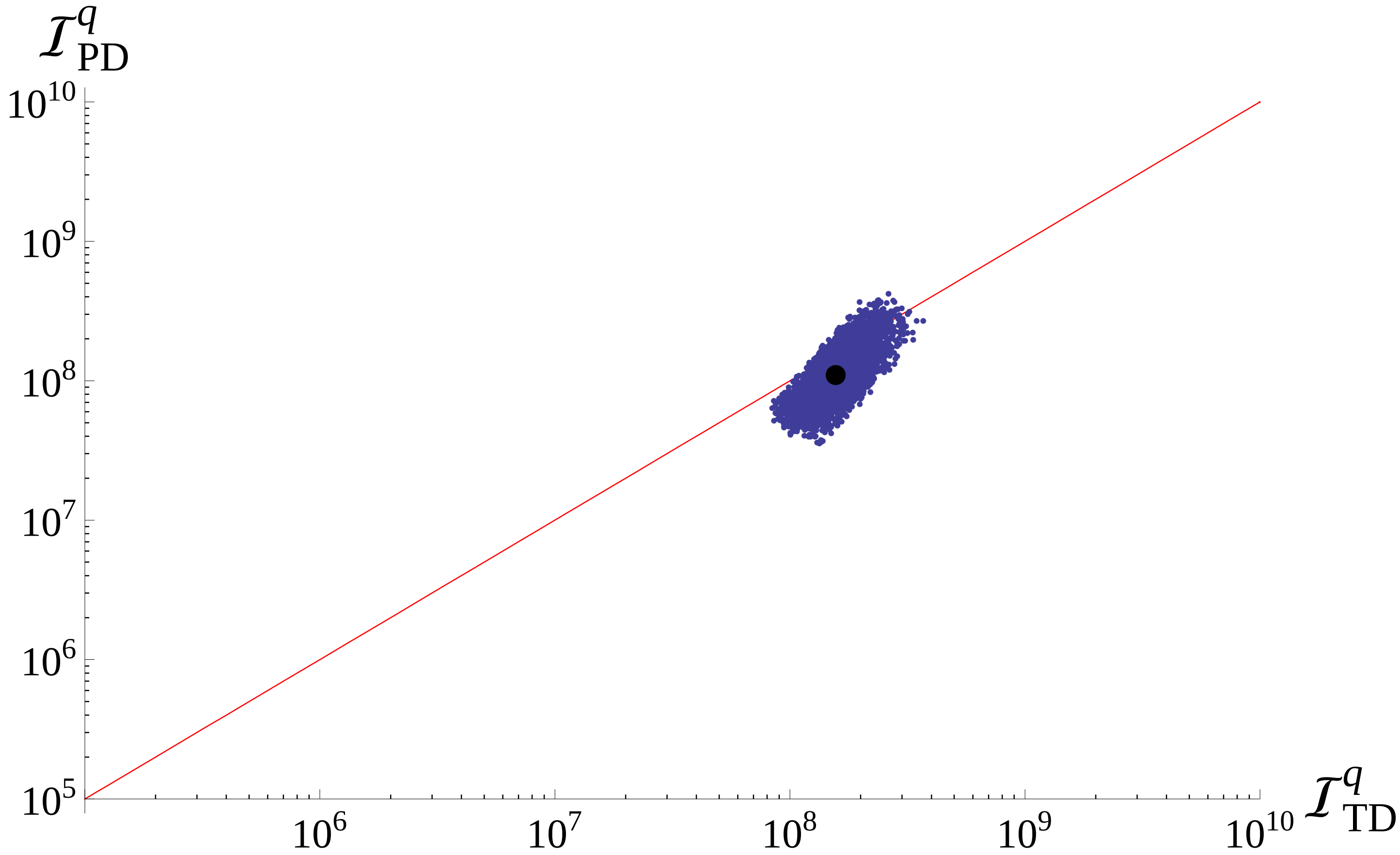
May be evaluated in terms of the conventional mass and CKM observables:

$$\mathcal{I}^q_{TD} = \sum_{\alpha\neq\beta\neq\gamma, i\neq j\neq k} \left(\frac{m_\alpha^2}{m_\beta m_\gamma}\frac{m_i^2}{m_j m_k}\right)^{\frac{2}{3}} |V_{\alpha i}|^2 \simeq \left(\frac{m_t}{m_u}\frac{m_t}{m_c}\frac{m_b}{m_d}\frac{m_b}{m_s}\right)^{\frac{2}{3}} \cos^2\theta_{23} \sim 10^8;$$

$$\mathcal{I}^q_{PD} = \sum_{\alpha\neq\beta\neq\gamma, i\neq j\neq k} \left(\frac{m_\beta m_\gamma}{m_\alpha^2}\frac{m_j m_k}{m_i^2}\right)^{\frac{2}{3}} |V_{\alpha i}|^2 \simeq \left(\frac{m_t}{m_u}\frac{m_c}{m_u}\frac{m_b}{m_d}\frac{m_s}{m_d}\right)^{\frac{2}{3}} \cos^2\theta_{12} \sim 10^8,$$

evaluated at leading order in small mass ratios.

Experimental Values of $\mathcal{I}^q_{TD}$ and $\mathcal{I}^q_{PD}$



Evaluated using experimental values of quark masses and mixings (renormalised to the weak scale by Xing et al., PRD 77, 113016 (2008)).

An Application - Coincident Values?

- $\mathcal{I}^q_{TD}$ and $\mathcal{I}^q_{PD}$ are independent combinations of quark masses and mixings
- Could have taken any values in nature.
- Experimentally:

$$\frac{\mathcal{I}^q_{PD}}{\mathcal{I}^q_{TD}} \approx \left(\frac{m_c^2}{m_t m_u}\frac{m_s^2}{m_b m_d}\right)^{\frac{2}{3}} \frac{\cos^2\theta_{23}}{\cos^2\theta_{12}} = 0.7^{+1.1}_{-0.4}.$$

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- It's a mystery why their values are so similar
- Feature seems to require an unnatural level of fine-tuning - an accident of Nature?

What if $\mathcal{I}^q_{TD} = \mathcal{I}^q_{PD}$?

- Can be shown that if $\mathcal{I}^q_{TD} = \mathcal{I}^q_{PD}$, then the spectrum of $\mathcal{UD}$ is exactly geometric!

$$\mathcal{I}^q_{TD} = \mathcal{I}^q_{PD} \implies \frac{\lambda_2}{\lambda_1} = \frac{\lambda_3}{\lambda_2}$$

- Eigenvalue ratios are also one-loop evolution invariants $\implies$ geometric hierarchy at all energy scales.
- Data are thus consistent with spectrum of $\mathcal{UD}$ being geometric at all scales
- Suggestive that some New Physics at a high scale requires a geometric spectrum for $\mathcal{UD}$.

Readily Generalised to the Leptons

For the leptons, in Dirac ν case, $\mathcal{I}^{q}_{TD} \to \mathcal{I}^{\ell}_{TD}$, and $\mathcal{I}^{q}_{PD} \to \mathcal{I}^{\ell}_{PD}$ with $\mathcal{U} \to \mathcal{N}$ and $\mathcal{D} \to \mathcal{L}$, $\gamma_u \to \gamma_\nu$ and $\gamma_d \to \gamma_\ell$.

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Also find two evolution invariants linking quarks, leptons and gauge couplings:

$$\mathcal{I}^{ql}_{\rm prod} \equiv \frac{\mathrm{Det}(\mathcal{U}\mathcal{D})}{\mathrm{Det}(\mathcal{N}\mathcal{L})} g_1^{-\frac{96}{41}} g_3^{-\frac{96}{7}}$$

$$\mathcal{I}^{ql}_{\rm comm} \equiv \frac{\mathrm{Det}^3[\mathcal{U},\mathcal{D}]\mathrm{Det}[\mathcal{N},\mathcal{L}]}{\mathrm{Det}^3(\mathcal{U}\mathcal{D})\mathrm{Det}^{\frac{5}{4}}(\mathcal{N}\mathcal{L})} g_1^{-\frac{81}{82}} g_2^{\frac{81}{38}}.$$

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- More details in arXiv/1007.3810 [hep-ph]

BACKUP SLIDES

Charge $\frac{2}{3}$ Quark Masses - Eigenvalues of $\mathcal{U}$

Given by its eigenvalue equation:

$$\lambda_i^3 - T_{\mathcal{U}} \cdot \lambda_i^2 + P_{\mathcal{U}} \cdot \lambda_i - D_{\mathcal{U}} = 0$$

where

$$T_{\mathcal{U}} = Tr(\mathcal{U}); \quad P_{\mathcal{U}} = \frac{1}{2}(Tr^2(\mathcal{U}) - Tr(\mathcal{U}^2)); \quad D_{\mathcal{U}} = Det(\mathcal{U})$$

But RGEs of the coefficients are complicated:

$$\frac{dT_{\mathcal{U}}}{dt} = 2\gamma_u T_{\mathcal{U}} + 3(T_{\mathcal{U}}^2 - 2P_{\mathcal{U}} - Tr(\mathcal{U}\mathcal{D}))$$

$$\frac{dP_{\mathcal{U}}}{dt} = 4\gamma_u P_{\mathcal{U}} + 3P_{\mathcal{U}}(T_{\mathcal{U}} - T_{\mathcal{D}}) + 3D_{\mathcal{U}}(Tr(\mathcal{U}^{-1}\mathcal{D}) - 3)$$

$$\frac{dD_{\mathcal{U}}}{dt} = 3D_{\mathcal{U}}(2\gamma_u + T_{\mathcal{U}} - T_{\mathcal{D}})$$

So, evolution of the eigenvalues of $\mathcal{U}$ depends in a complicated way on the eigenvalues of $\mathcal{U}$, $\mathcal{D}$ and on elements of V_{CKM}. Similar conclusion for e/values of $\mathcal{D}$.