# Quantum Entropy Function

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# Plan

**By now there are many examples in string theory where the correspondence between black hole entropy and statistical entropy has been tested for extremal BPS black holes.**

$$S_{BH}(Q) = S_{stat}(Q)$$

$$S_{BH} = A/4G_N, \qquad S_{stat} = \ln d_{micro}$$

**Initial tests were carried out for black holes carrying large charges for which the computation simplifies on both sides.**

**This suggests that in the large charge limit a black hole represents an ensemble of microstates whose total number is given by exp$[S_{BH}]$.**

**What happens beyond the large charge limit?**

**On the microscopic side we can, in principle, count states to arbitrary accuracy.**

**Is the microscopic description more fundamental, and black holes only capture some average properties in the limit of large size?**

**Or, does a black hole contain complete information about the ensemble?**

**For example:**

**1. Do black holes encode systematically corrections to the entropy due to finite size effect?**

**2. Are black holes capable of computing the distribution of global quantum numbers among the microstates?**

Quantum entropy function is an attempt to answer these questions in the affirmative.

– provides an algorithm for computing black hole entropy, distribution of global charges, etc. to arbitrary accuracy.

Many of the results have been tested by independent calculations on the microscopic side.

# Quantum entropy function

**Let us denote by $d_{hor}$ the degeneracy associated with the black hole horizon.**

**In the leading order**

$$d_{hor} = \exp[S_{BH}]$$

**In string theory this receives two types of corrections.**

1. **Higher derivative ($\alpha'$) corrections in classical string theory.**
2. **Quantum ($g_s$) corrections.**

**Of these the $\alpha'$ corrections are captured by Wald's modification of the Bekenstein-Hawking formula.**

**Thus in classical string theory $d_{hor} = \exp[S_{wald}]$**

## Quantum corrections:

**What about quantum corrections?**

**Naive guess: apply Wald's formula again, but replacing the classical action by the 1PI action.**

**This will again give a simple algebraic method for computing the entropy.**

**This prescription is not complete since the 1PI action typically has non-local contribution due to massless states propagating in the loops.**

We shall now turn to the full quantum computation of $d_{hor}$ from the macroscopic side.

The main tool: $AdS_2/CFT_1$ correspondence.

– uses the observation that the near horizon geometry of an extremal black hole always has the form of $AdS_2 \times K$.

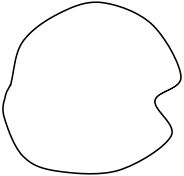
# Steps for computing $d_{hor}$

**1. Consider the euclidean $AdS_2$ metric:**

$$ds^2 = v\left((r^2-1)d\theta^2 + \frac{dr^2}{r^2-1}\right), \quad 1 \le r < \infty, \quad \theta \equiv \theta + 2\pi$$
$$= v(\sinh^2\eta\, d\theta^2 + d\eta^2), \quad r \equiv \cosh\eta, \quad 0 \le \eta < \infty$$

**Regularize the infinite volume of $AdS_2$ by putting a cut-off $r \le r_0 f(\theta)$ for some smooth periodic function $f(\theta)$.**

**This makes the $AdS_2$ boundary have a finite length L.**



**2. Define $Z_{AdS_2}$ to be the partition function of string theory in the near horizon $AdS_2 \times K$ geometry.**

**3. By $AdS_2/CFT_1$ correspondence:**

$$Z_{AdS_2} = Z_{CFT_1}$$

$$Z_{CFT_1} = Tr(e^{-LH}) = d_0\, e^{-L\,E_0}$$

**H: Hamiltonian of dual $CFT_1$ at the boundary of $AdS_2$.**

**$(d_0, E_0)$: (degeneracy, energy) of the states of $CFT_1$.**

$$Z_{AdS_2} = Z_{CFT_1} = d_0\, e^{-L\, E_0}$$

**What is $CFT_1$?**

– must be the quantum mechanics obtained by taking the infrared limit of the brane system describing the black hole.

**This consists of a finite dimensional Hilbert space, consisting of the ground states of the brane system in a given charge sector.**

Thus $d_0$ is the number of quantum states of the extremal black hole.

**This suggests that we identify $d_{hor}$ with $d_0$.**

4. Thus we can define $d_{hor}$ by expressing $Z_{AdS_2}$ as

$$Z_{AdS_2} = e^{CL} \times d_{hor} \quad \text{as } L \to \infty$$

C: A constant

$d_{hor}$: 'finite part' of $Z_{AdS_2}$

With this definition $d_{hor}$ calculates $d_0$, i.e. the degeneracy of the dual $CFT_1$.

**Consistency check:**

$d_{hor}$ gives us back $\exp[S_{wald}]$ in the classical limit.

# Global charge distribution

**Suppose the theory has a global $\mathbb{Z}_N$ symmetry generated by g.**

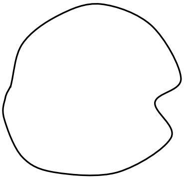
**Can we calculate the weighted degeneracy**

$$\mathrm{Tr}(g)?$$

**By calculating $\mathrm{Tr}(g^k)$ for all k we can find the distribution of g quantum numbers among the microstates of the black hole.**

**By following the logic of AdS/CFT correspondence we find that $\mathrm{Tr}_{hor}(g)$ is given by the finite part of a twisted partition function**

**– the path integral is to be carried out over fields satisfying g twisted boundary condition under $\theta \to \theta + 2\pi$.**



**Since euclidean $AdS_2$ is contractible the original near horizon geometry is no longer a valid saddle point in the path integral.**

**The leading saddle point comes from a $\mathbb{Z}_N$ orbifold of the original near horizon geometry.**

**In the classical limit one gets**

$$\mathbf{Tr(g)} \sim \exp[\mathbf{S_{wald}}/\mathbf{N}]$$

# Tests

Many of the predictions of quantum entropy function have been verified using the exact microscopic results in a class of $\mathcal{N} = 4$ supersymmetric string theories.

In these theories the degeneracies and weighted degeneracies of black hole microstates can be calculated exactly by representing them as a configuration of branes.

The main obstruction: Calculating the quantum entropy function by evaluating the path integral of string theory in the near horizon geometry.

Nevertheless to whatever extent the latter has been calculated the result always agrees with the microscopic results.

**These agreements include:**

1. **The leading contribution.**

2. **First subleading corrections.**

3. **Logarithmic corrections ($\propto \ln A$)**

4. **Prediction for Tr(g)**

**etc.**

# Conclusions

**Quantum gravity in the near horizon geometry contains detailed information about not only the total number of microstates. but also finer details *e.g.* the $\mathbb{Z}_N$ quantum numbers carried by the microstates.**

**Thus at least for extremal black holes there seems to be an exact duality between**

**Gravity description $\Leftrightarrow$ Microscopic description**

**The gravity description contains as much information as the microscopic description, but in quite different way.**