Physics of the Cosmic Microwave Background

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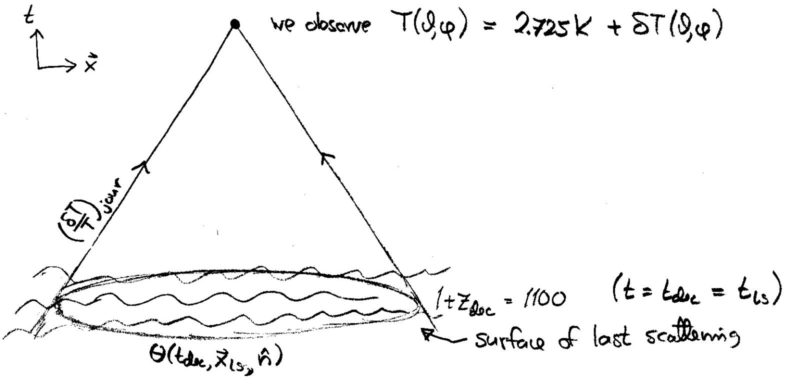
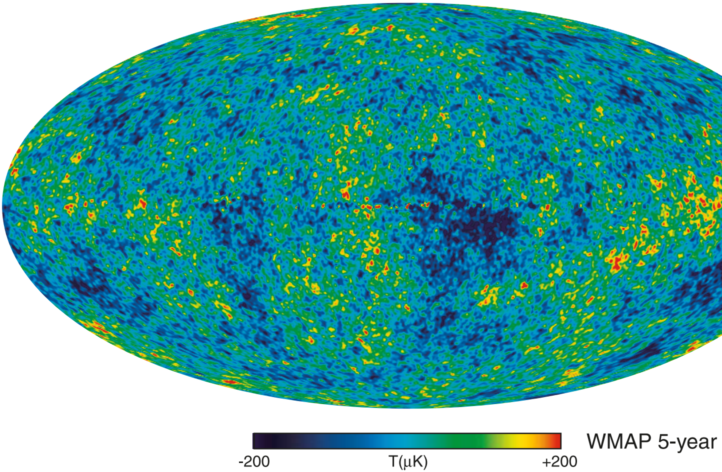


Figure: The observed CMB temperature anisotropy gets a contribution from the last scattering surface, $(\delta T/T)_{\mathrm{intr}} = \Theta(t_*, \vec{x}_{\mathrm{ls}}, \hat{q})$ and from along the photon's journey to us, $(\delta T/T)_{\mathrm{jour}}$.

The CMB anisotropy is small

RMS temperature variation $\sim 100\mu$K



Relative variation $\sim 4 \times 10^{-5}$
1st order perturbation theory
around a homogeneous and isotropic model of the Universe
(background model)
background + perturbation

$$\rho = \bar{\rho} + \delta\rho = (1 + \delta)\bar{\rho}$$

Background universe

Friedmann-Robertson-Walker (flat case)

$$ds^2 = -dt^2 + a(t)^2 \left(dx^2 + dy^2 + dz^2\right)$$

$a(t)$ the scale factor describes the expansion of the universe
In the early universe (radiation dominated) $a \propto t^{1/2}$
Later (matter dominated) $a \propto t^{2/3}$
Late times: dark energy (?) causes the expansion to accelerate

$H(t) = \dfrac{1}{a}\dfrac{da}{dt}$ Hubble parameter gives the expansion rate

The "perturbed" (real) universe

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(1 - 2\Psi)(dx^2 + dy^2 + dz^2)$$

$\Phi(x, y, z)$	Newtonian potential
$\Psi(x, y, z)$	Curvature perturbation

Matter and energy components

$$\rho = \rho_\gamma + \rho_\nu + \rho_{\rm cdm} + \rho_b = \sum \rho_i$$

$$p = p_\gamma + p_\nu + p_{\rm cdm} + p_b = \rho_\gamma/3 + \rho_\nu/3 + p_b = \sum p_i$$

$$\rho_i = (1 + \delta_i)\bar{\rho}_i$$

$$p_i = \bar{p}_i + \delta p_i$$

Fluid perturbation variables: δ_i, δp_i, $\vec{v}_i$
If ρ is perfect fluid $\Rightarrow \Phi = \Psi$
(get $\sim 10\%$ differences due to neutrinos)
Fluid description is not enough for photons (and neutrinos)

Photon distribution function

$$f(t, \vec{x}, \vec{q})$$

$$dN = \frac{2}{(2\pi)^3} f dV d^3 q$$

Here $\vec{q} \equiv q\hat{q}$ is the photon momentum
In the background model, photons have the blackbody spectrum

$$\bar{f}(t, \vec{q}) = \frac{1}{e^{q/T(t)} - 1}$$

In the perturbed universe

$$f = \bar{f} + \delta f = \frac{1}{\exp\left\{\frac{q}{T(t)[1+\Theta(t,\vec{x},\vec{q})]}\right\} - 1}$$

any function $f(t, \vec{x}, \vec{q})$ can be written in this form, but the physics result is that to 1st order, Θ does not develop any q-dependence!

Brightness function

$$\Theta = \Theta(t, \vec{x}, \hat{q})$$

depends only on the photon direction, not on the photon energy (in general, this holds for massless particles)

$$\delta_\gamma = 4\Theta_0, \quad \text{where} \quad \Theta_0(t, \vec{x}) \equiv \frac{1}{4\pi} \int \Theta(t, \vec{x}, \hat{q}) d\Omega$$

$$\vec{v}_\gamma = 3\vec{\Theta}_1, \quad \text{where} \quad \vec{\Theta}_1(t, \vec{x}) \equiv \frac{1}{4\pi} \int \hat{q}\Theta(t, \vec{x}, \hat{q}) d\Omega$$

$$\Theta_2^{ij}(t, \vec{x}) \equiv \frac{1}{4\pi} \int \left(\hat{q}^i \hat{q}^j - \frac{1}{3}\delta_{ij} \right) \Theta(t, \vec{x}, \hat{q}) d\Omega$$

local monopole, dipole, and quadrupole of the photon perturbation

Boltzmann Equation

Liouville theorem: If there are no collisions, $f =$ const. along trajectory in phase space

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial q^i}\frac{dq^i}{dt} = 0$$

With collisions,

$$\frac{df}{dt} = C[f]$$

where $C[f]$ is the collision term.

In the curved spacetime, photons travel on lightlike geodesics.

$$\frac{dx^i}{dt} = \frac{\hat{q}^i}{a}$$

$$\text{geodesic equation} \Rightarrow \qquad \frac{dq^i}{dt}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{q}^i}{a}\frac{\partial f}{\partial x^i} + p\frac{\partial f}{\partial p}\left[-H - \frac{\hat{q}^i}{a}\frac{\partial \Phi}{\partial x^i} + \frac{\partial \Psi}{\partial t}\right] = C[f]$$

$$\frac{df}{dt} = \underbrace{\frac{\partial f}{\partial t} + \frac{\hat{q}^i}{a}\frac{\partial f}{\partial x^i}}_{kinematics} + p\frac{\partial f}{\partial p}\left[\underbrace{-H}_{expansion} - \underbrace{\frac{\hat{q}^i}{a}\frac{\partial \Phi}{\partial x^i} + \frac{\partial \Psi}{\partial t}}_{\text{spacetime perturbations}}\right] = C[f]$$

Separate this into a background equation

$$\frac{d\bar{f}}{dt} = \frac{\partial \bar{f}}{\partial t} - Hp\frac{\partial \bar{f}}{\partial p} = 0$$

(the effect of collisions can be ignored at the background level)

$$\ldots \Rightarrow T \propto 1/a$$

and the perturbation equation

$$\ldots \Rightarrow \frac{\partial \Theta}{\partial t} + \frac{\hat{q}^i}{a}\frac{\partial \Theta}{\partial x^i} + \frac{\hat{q}^i}{a}\frac{\partial \Phi}{\partial x^i} - \frac{\partial \Psi}{\partial t} = C[\Theta]$$

Collision term: Thomson scattering

Photons scatter on electrons

$$\frac{d\sigma}{d\Omega} = \frac{\sigma_T}{4\pi}\frac{3}{4}\left(1 + \cos^2\theta\right)$$

$$\frac{\partial\Theta}{\partial t} + \frac{\hat{q}^i}{a}\frac{\partial\Theta}{\partial x^i} + \frac{\hat{q}^i}{a}\frac{\partial\Phi}{\partial x^i} - \frac{\partial\Psi}{\partial t} = n_e\sigma_T\left[\Theta_0 - \Theta(\hat{q}) + \hat{q}\cdot\vec{v}_b + \frac{3}{4}\hat{q}^i\hat{q}^j\Theta_2^{ij}\right]$$

This is called the **Brightness equation**

(Boltzmann equation for photons)

Recombination

The early universe was filled with plasma: electrons, ions, photons, (neutrinos, CDM particles).
Ar $t \sim 380000$ yr, when $T \sim 4000$ K, electrons and ions formed atoms (mainly hydrogen).
The density of free electrons n_e dropped by a large factor.
Approximation: instantaneous recombination (photon decoupling) at $t = t_*$
$t < t_*$: tight coupling (n_e large)

$$\Theta(\hat{q}) = \Theta_0 + \hat{q} \cdot \vec{v}_b \quad \Rightarrow \quad \vec{v}_\gamma \equiv 3\Theta_1 = \vec{v}_b; \quad \Theta_2^{ij} = 0$$

$t > t_*$: no collisions (n_e small)

$$\underbrace{\frac{\partial \Theta}{\partial t} + \frac{\hat{q}}{a}\frac{\partial \Theta}{\partial x^i}}_{\frac{d\Theta}{dt}} + \underbrace{\frac{\hat{q}}{a}\frac{\partial \Phi}{\partial x^i}}_{\frac{d\Phi}{dt} - \frac{\partial \Phi}{\partial t}} - \frac{\partial \Psi}{\partial t} = 0 \quad \Rightarrow \quad \frac{d}{dt}(\Theta + \Phi) = \frac{\partial \Phi}{\partial t} + \frac{\partial \Psi}{\partial t}$$

Line-of-sight integration

Integrate

$$\frac{d}{dt}(\Theta + \Phi) = \frac{\partial \Phi}{\partial t} + \frac{\partial \Psi}{\partial t}$$

along the photon path from there $(t_*, \vec{x}_{ls})$ to here $(t_0, \vec{x}_{\rm obs})$:

$$\begin{aligned}
&\underbrace{\Theta(t_0, \vec{x}_{\rm obs}, \hat{q})}_{\frac{\delta T}{T}(\theta,\phi)} + \underbrace{\Phi(t_0, \vec{x}_{\rm obs})}_{\textit{constant for us}} \\
&= (\Theta + \Phi)(t_*, \vec{x}_{ls}, \hat{q}) + \int_{t_*}^{t_0} \left(\frac{\partial \Phi}{\partial t} + \frac{\partial \Psi}{\partial t} \right) dt \\
&= \underbrace{\underbrace{\Theta_0(t_*, \vec{x}_{ls})}_{\frac{1}{4}\delta_\gamma} + \Phi(t_*, \vec{x}_{ls})}_{\textit{monopole term}} \underbrace{\underbrace{+\hat{q} \cdot \vec{v}_{b\gamma}}_{-\hat{n}\cdot\vec{v}_{b\gamma}}}_{\textit{dipole term}} + \underbrace{\int_{t_*}^{t_0} \left(\frac{\partial \Phi}{\partial t} + \frac{\partial \Psi}{\partial t} \right) dt}_{\textit{Integrated Sachs-Wolfe effect}}
\end{aligned}$$

monopole term = effective temperature perturbation
dipole term = Doppler effect

The full thing

Φ, Ψ affected by all energy components ρ_b, ρ_ν, ρ_b, ρ_{cdm}
Need their perturbation equations also & the GR equations for Φ, Ψ
Everything starts from primordial perturbations (initial values for perturbation eqs.)
apparently produced by some random process (quantum fluctuations during inflation) in the very early universe

Adiabatic primordial perturbations

Simplest inflation models: one independent quantity: the inflaton field ϕ

The homogeneous background value $\bar{\phi}(t)$ rolls slowly down its potential $V(\phi)$

All perturbations originate from $\delta\phi \quad \Rightarrow$ adiabatic perturbations

$$\delta\left(\frac{n_i}{n_\gamma}\right) = 0 \quad \Rightarrow \quad \frac{\delta n_i}{n_i} = \underbrace{\frac{\delta n_\gamma}{n_\gamma}}_{3\frac{\delta T}{T}} = \frac{3}{4}\underbrace{\frac{\delta \rho_\gamma}{\rho_\gamma}}_{4\frac{\delta T}{T}}$$

For baryons, CDM, $\rho_i = m_i n_i \quad \Rightarrow \quad \underbrace{\frac{\delta\rho_i}{\rho_i}}_{\delta_i} = \frac{\delta n_i}{n_i}$

Thus $\delta_b = \delta_c \equiv \delta_m = \frac{3}{4}\delta_\gamma$ initially

Outside horizon

Horizon $\equiv$ distance of causal interaction within cosmological time scale $= H^{-1}$, the Hubble distance

After inflation, all scales of interest are "beyond horizon" $\Rightarrow$ they do not evolve

"Superhorizon" perturbations most naturally described in terms of spacetime curvature: "comoving curvature perturbation" $\mathcal{R}(t, \vec{x})$

Inflation produces close to scale-independent ($n = 1$) primordial perturbations

$$\mathcal{P}_{\mathcal{R}}(k) \equiv \frac{V}{2\pi^2} k^3 \underbrace{\times \langle |\mathcal{R}_{\vec{k}}|^2 \rangle}_{expectation\ value} = \frac{1}{24\pi^2 M_{Pl}^4} \frac{V(\phi_x)}{\epsilon(\phi_x)} \approx const. \equiv A^2$$

$$or \quad A^2 \left(\frac{k}{k_p} \right)^{n-1} \qquad n - 1 = -6\epsilon + 2\eta$$

Entering horizon

After inflation, as the universe gets older, the horizon H^{-1} grows, and encompasses larger scales ("scales enter the horizon")
At photon decoupling $t = t_*$, $H^{-1} \approx 200$ Mpc, about 1° on the CMB sky
At angles $> 1^\circ$ we see superhorizon perturbations

Large scales: Still outside horizon at decoupling (t_*)

$$\begin{aligned} \tfrac{1}{4}\delta_\gamma &= \tfrac{1}{3}\delta_m \sim \tfrac{1}{3}\delta \qquad (\textit{assume matter domination}) \\ \vec{v}_{b\gamma} &\sim 0 \end{aligned}$$

$$\frac{\delta T}{T}(\theta,\phi) = \tfrac{1}{3}\delta + \Phi + \int \left(\frac{\partial \Phi}{\partial t} + \frac{\partial \Psi}{\partial t} \right) dt$$

Friedmann $H^2 = \frac{8\pi G}{3}\bar{\rho}$

Newton $\nabla^2 \Phi = 4\pi G \delta\rho = \underbrace{4\pi G \bar{\rho}}_{\frac{3}{2}H^2} \cdot \delta \quad \Rightarrow \quad \delta_{\vec{k}} = -\frac{2}{3}\left(\frac{k}{H}\right)^2 \Phi_{\vec{k}}$

GR $\quad \rightarrow \quad \delta_{\vec{k}} = -\left[2 + \frac{2}{3}\left(\frac{k}{H}\right)^2\right]\Phi_{\vec{k}}$

For superhorizon ($k \ll H$) scales, $\delta \approx -2\Phi$

Thus $\frac{1}{3}\delta + \Phi \approx -\frac{2}{3}\Phi + \Phi = \frac{1}{3}\Phi \qquad (\Phi = -\frac{3}{5}\mathcal{R},\ \delta = \frac{6}{5}\mathcal{R})$

Sachs–Wolfe effect:

$$\left(\frac{\delta T}{T}\right)_{obs} = \underbrace{\tfrac{1}{3}\Phi}_{ordinary} + \underbrace{\int \left(\frac{\partial \Phi}{\partial t} + \frac{\partial \Psi}{\partial t} \right) dt}_{integrated}$$

Angular Power Spectrum C_ℓ

$$\frac{\delta T}{T}(\theta,\phi) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta,\phi)$$

$$a_{\ell m} = \int d\Omega\, Y^*_{\ell m}(\theta,\phi) \frac{\delta T}{T}(\theta,\phi)$$

The $a_{\ell m}$ depend linearly (through linear physics of 1st order perturbation theory) on primordial perturbations

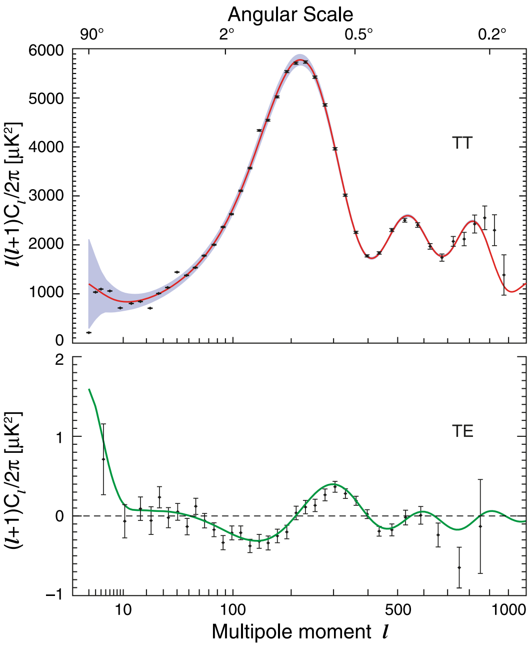
Result of random process $\Rightarrow$ predict statistical properties only

$$\langle a_{\ell m} \rangle = 0 \qquad \langle a_{\ell m} a^*_{\ell' m'} \rangle = 0 \text{ for } \ell \neq \ell' \text{ or } m \neq m'$$

$$\text{but} \quad C_\ell \equiv \langle |a_{\ell m}|^2 \rangle \neq 0 \quad \text{same for all } m$$

$\ell \sim$ structure at angular scale $180^\circ/\ell$ (half-wavelength)

$$\left\langle \left(\frac{\delta T}{T} \right)^2 \right\rangle = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell \qquad \text{(temperature variance)}$$



Angular Scale
90°
2°
0.5°
0.2°
6000
5000
4000
3000
2000
1000
0
TT
l(l+1)C_l/2π [μK²]
2
1
0
−1
TE
(l+1)C_l/2π [μK²]
10
100
500
1000
Multipole moment l

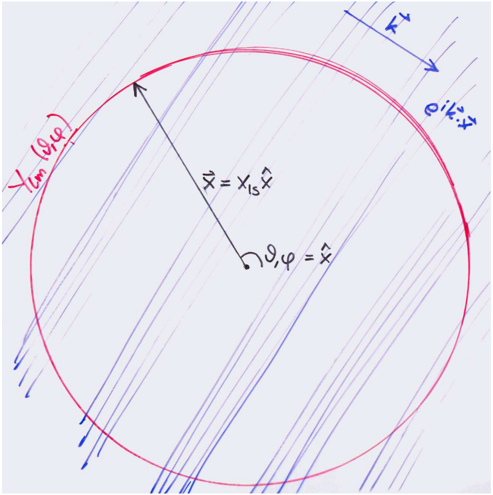


Figure: A plane wave intersecting the last scattering sphere.

C_ℓ from ordinary SW for large scales (small ℓ, where dominates)

$$
\begin{aligned}
a_{\ell m} &= \int Y^*_{\ell m}(\hat{x}) \frac{\delta T}{T}(\hat{x}) d\Omega_x \\
\frac{\delta T}{T}(\hat{x}) &= \tfrac{1}{3}\Phi(t_*, \vec{x}_{ls}) = -\tfrac{1}{5}\mathcal{R}(\vec{x}_{ls}) \\
\mathcal{R}(\vec{x}_{ls}) &= \sum_{\vec{k}} \mathcal{R}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}_{ls}} \\
e^{i\vec{k}\cdot\vec{x}_{ls}} &= 4\pi \sum_{\ell' m'} i^\ell j_\ell(k x_{ls}) Y_{\ell' m'}(\hat{x}) Y^*_{\ell' m'}(\hat{k})
\end{aligned}
$$

$$
\begin{aligned}
C_\ell &\equiv \frac{1}{2\ell+1} \sum_m \langle |a_{\ell m}|^2 \rangle = \ldots \\
&= \frac{4\pi}{25} \sum_{\vec{k}} \langle |\mathcal{R}_{\vec{k}}|^2 \rangle j_\ell(kx)^2 \\
&= \frac{4\pi}{25} \int \frac{dk}{k} \mathcal{P}_{\mathcal{R}}(k) j_\ell(kx)^2
\end{aligned}
$$

For $n = 1$ ($\mathcal{P}_{\mathcal{R}}(k) = const.$),

$$C_\ell = \frac{4\pi}{25}\mathcal{P}_{\mathcal{R}} \int \frac{dk}{k} j_\ell(kx)^2 = \frac{\mathcal{P}_{\mathcal{R}}}{25} \cdot \frac{2\pi}{\ell(\ell+1)}$$

$$\frac{\ell(\ell+1)}{2\pi} C_\ell = \frac{\mathcal{P}_{\mathcal{R}}}{25} = \frac{1}{600\pi^2 M_{Pl}^4} \frac{V(\phi_x)}{\epsilon(\phi_x)} \approx \frac{1000\mu\mathrm{K}^2}{2.7\mathrm{K}^2} \approx 1.3 \times 10^{-10}$$

$$\frac{V(\phi_x)}{\epsilon(\phi_x)} \approx 8 \times 10^{-7} M_{Pl}^4 \approx (0.03 M_{Pl})^4$$

gives upper limit to inflation scale:
$V(\phi_x)^{1/4} < 0.03 M_{Pl} = 7 \times 10^{16}$ GeV ($\epsilon \ll 1$)

C_ℓ for larger ℓ (smaller scales)

$$\frac{\delta T}{T}(\theta,\phi) = \Theta_0 + \Phi - \hat{n}\cdot\vec{v} + \int\left(\frac{\partial\Phi}{\partial t} + \frac{\partial\Psi}{\partial t}\right)$$

Smaller scales entered horizon before t_*
CDM perturbations grow $\Rightarrow$ dominate Φ
baryon+photon perturbations oscillate

$$\Theta_{0\vec{k}} + (1+R)\Phi_{\vec{k}} \propto \cos c_s kt \qquad R \equiv \frac{3}{4}\frac{\rho_b}{\rho_\gamma}$$

$$c_s^2 = \frac{1}{3}\frac{1}{1+R} \quad \text{sound speed}$$

Expansion: $c_s t \rightarrow r_s(t) \equiv \int_0^t \frac{c_s(t)}{a(t)}dt$ sound horizon

$$(\Theta_0 + \Phi)_{\vec{k}}(t_*) = -R\Phi_{\vec{k}}(t_*) + A_{\vec{k}}\cos kr_s(t_*)$$

Maximal at scales k: $kr_s = m\pi$
Strong structure at multipoles $\ell = kd_A(t_*) = m\pi\frac{d_A}{r_s} \equiv m\ell_A$
where $\ell_A \equiv \pi\frac{d_A(t_*)}{r_s(t_*)} \equiv \frac{\pi}{\theta_s}$

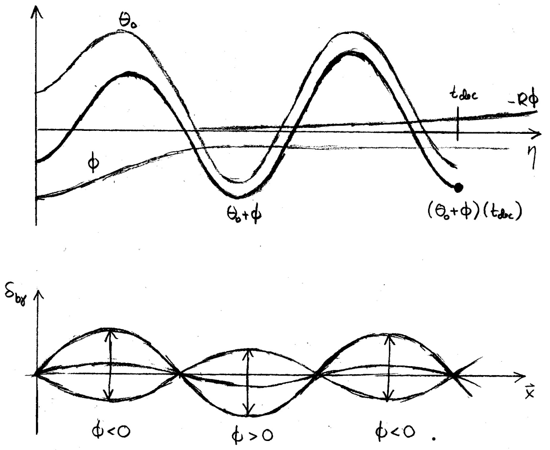


Figure: Acoustic oscillations. The top panel shows the time evolution of the Fourier amplitudes $\Theta_{0\vec{k}}$, $\Phi_{\vec{k}}$, and the effective temperature $\Theta_{0\vec{k}} + \Phi_{\vec{k}}$. The Fourier mode shown corresponds to the fourth acoustic peak of the C_ℓ spectrum. The bottom panel shows $\delta_{b\gamma}(\vec{x})$ for one Fourier mode as a function of position at various times (maximum compression, equilibrium level, and maximum decompression).

$d_A(t_*)$ angular diameter distance to last scattering
ℓ_A acoustic scale in multipole space
θ_s sound horizon angle
The Doppler effect $-\hat{n} \cdot \vec{v}$ oscillates too, but off-phase
C_ℓ is quadratic in $\delta T/T \quad \Rightarrow$ has also cross terms of $\Theta_0 + \Phi$, $-\hat{n} \cdot \vec{v}$, and $\int \left(\frac{\partial \Phi}{\partial t} + \frac{\partial \Psi}{\partial t}\right)$
Diffusion damping: Actually photon decoupling is not instantaneous $\Rightarrow$ photon diffusion partially erases photon perturbations at scales comparable or smaller than the photon mean free path

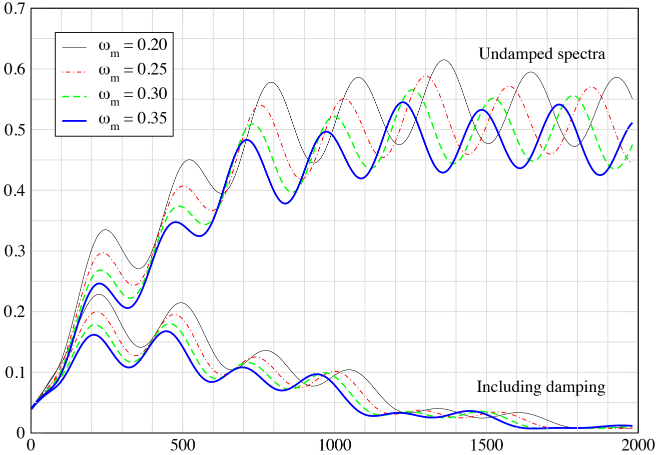
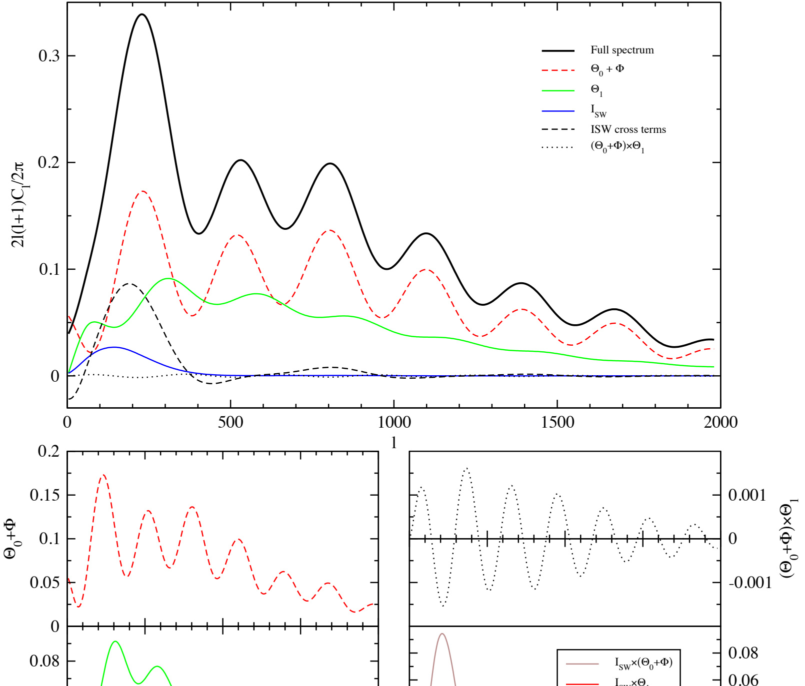


Figure: The angular power spectrum C_ℓ, calculated both with and without the effect of diffusion damping. The spectrum is given for four different values of ω_m, with $\omega_b = 0.01$. (This is a rather low value of ω_b, so $\ell_D < 1500$ and damping is quite strong.) Figure and calculation by R. Keskitalo.



Full spectrum
Θ0 + Φ
Θ1
ISW
ISW cross terms
(Θ0+Φ)×Θ1
2l(l+1)Cl/2π
l
Θ0+Φ
(Θ0+Φ)×Θ1
ISW×(Θ0+Φ)

Figure: The full C_ℓ spectrum calculated for the cosmological model $\Omega_0 = 1$, $\Omega_\Lambda = 0$, $\omega_m = 0.2$, $\omega_b = 0.03$, $A = 1$, $n = 1$, and the different contributions to it. (The calculation involves some approximations which allow the description of C_ℓ as just a sum of these contributions and is not as accurate as a CMBFAST or CAMB calculation.) Here Θ_1 denotes the Doppler effect. Figure and calculation by R. Keskitalo.