Cosmological inflation

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Basic definitions

Homogeneous and isotropic Friedmann-Robertson-Walker universe

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \ , \quad k = \pm 1, 0 \qquad (1)$$

Note that r is unitless and $a(t)$ has unit of length.

Energy-momentum tensor is

$$\begin{aligned} T_{\mu\nu} &= \rho u_\mu u_\nu + p(g_{\mu\nu} + u_\mu u_\nu) \\ &\Rightarrow T_{00} = \rho,\ T_{ii} = p \end{aligned} \qquad (2)$$

in the rest frame of the fluid ($u = (1, 0)$).

Einstein equations are:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{3} T_{\mu\nu}$$

Einstein equations can be derived from the action

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2} M_P^2 R + \mathcal{L}_{\text{matter}} \right) \tag{3}$$

where $g = det g_{\mu\nu}$. We require $\delta S = 0$. Heuristically

$$\frac{\delta\sqrt{-g}}{\delta g_{\mu\nu}} = -\frac{\delta(\Pi\lambda_a)^{1/2}}{\delta\lambda_b} = \frac{1}{2\lambda_b}(-g)^{-1/2} = \frac{1}{2} g^{\mu\nu}(-g)^{-1/2}$$

Since $\delta g_{\mu\nu} g^{\mu\nu} = 0$, we find

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2} g_{\mu\nu}(-g)^{-1/2} \quad . \tag{4}$$

Less straightforward to show that

$$\frac{\delta R}{\delta g^{\mu\nu}} = +R_{\mu\nu} \tag{5}$$

Obviously

$$\delta R = \delta(g^{\mu\nu} R_{\mu\nu}) = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} \quad . \tag{6}$$

Now $R_{\mu\nu} = R_{\mu\nu}(\partial\Gamma, \Gamma)$ with Γ the connection, and one can show that $\delta R_{\mu\nu} \propto \nabla\Gamma$, which is a total derivative and gives zero when integrated.

The energy-momentum tensor is simply

$$T_{\mu\nu} = -2\frac{\delta(\sqrt{-g}\mathcal{L}_{\text{matter}})}{\delta g^{\mu\nu}}$$

FRW Universe

Einstein equations $\Rightarrow$

$$\begin{aligned} H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad \text{(from } T_{00}\text{)} \qquad (7) \\ \dot{\rho} + 3H(p+\rho) &= 0 \quad \text{(continuity eq)} \ . \qquad (8) \end{aligned}$$

Must be supplemented by the equation of state $p = w\rho$. The continuity equation contains the same information as the T_{ii}-equation.

radiation	$w = 1/2$
matter	$w = 0$
vacuum energy	$w = -1$
stiff fluid	$w = 1$
phantom	$w < -1$

Table 1: Equations of state

Problem with curvature

If $k = 0$, then

$$H^2 = \frac{8\pi G}{3}\rho \equiv \frac{8\pi G}{3}\rho_c \Rightarrow \rho_c = \frac{3H^2}{8\pi G} \tag{9}$$

so that

$$\rho_c = \rho - \frac{3k}{8\pi a^2} \ . \tag{10}$$

We define

$$\Omega \equiv \rho/\rho_c \tag{11}$$

so that

$$\Omega - 1 = \frac{3k}{8\pi\rho_c a^2} \tag{12}$$

Thus

$$\frac{(\Omega - 1)_{\text{now}}}{(\Omega - 1)} = \frac{H^2 a^2}{H_{\text{now}}^2 a_{\text{now}}^2} \ . \tag{13}$$

Assume $a(t) \propto t^p$ so that $H \propto p/t$. For (adiabatic) radiation domination, $p = 1/2$, matter domination $p = 2/3$. Then

$$(\Omega - 1) = \left(\frac{t}{t_{\text{now}}} \right)^{2(1-p)} (\Omega - 1)_{\text{now}} \ . \tag{14}$$

Example: at the electroweak phase transition $t_{EW} = 10^{-11}$ s. Assume $t_{\text{now}} = 13.7 \times 10^9 \ yrs = 4.3 \times 10^{17} s$, plus radiation domination before $t = 380000 \ yrs$ followed by matter domination. Then

$$\left(\frac{t}{t_{\text{now}}} \right)^{2(1-p)} = \left(\frac{380000}{t_{\text{now}}} \right)^{2/3} \left(\frac{10^{-11}}{3800000} \right) = 7.5 \times 10^{-28} \ . \tag{15}$$

Fine-tuning of initial $\Omega \simeq 1$!

Solution: de Sitter space

If initially at $t = t_0$ the energy of the vacuum $\rho_\Lambda \gg \rho_m \simeq const.$, then $H^2 = 8\pi G\rho_\Lambda/3 = H_0^2 = const$ and

$$\begin{aligned} \dot{a}^2 &= k + H_0^2 a^2 \to H_0^2 a^2, \quad t \gg t_0 \\ &\Rightarrow \quad a(t) = a_0 \exp(H_0 t) \\ &\Rightarrow \quad (\Omega - 1) \propto \exp(-2H_0 t) \to 0 \ . \end{aligned} \tag{16}$$

Problem of acausal CMB

Photons travel along geodesics

$$ds^2 = 0 \ . \tag{17}$$

Take flat ($\Omega = 1$) FRW space for simplicity. Then

$$ds^2 = dt^2 - a^2 dx^2 = 0 \tag{18}$$

and the physical distance photon travels is

$$d(t) = a(t) \int dx = a(t) \int_0^t dt a^{-1} = \frac{t^p t^{1-p}}{1-p} \tag{19}$$

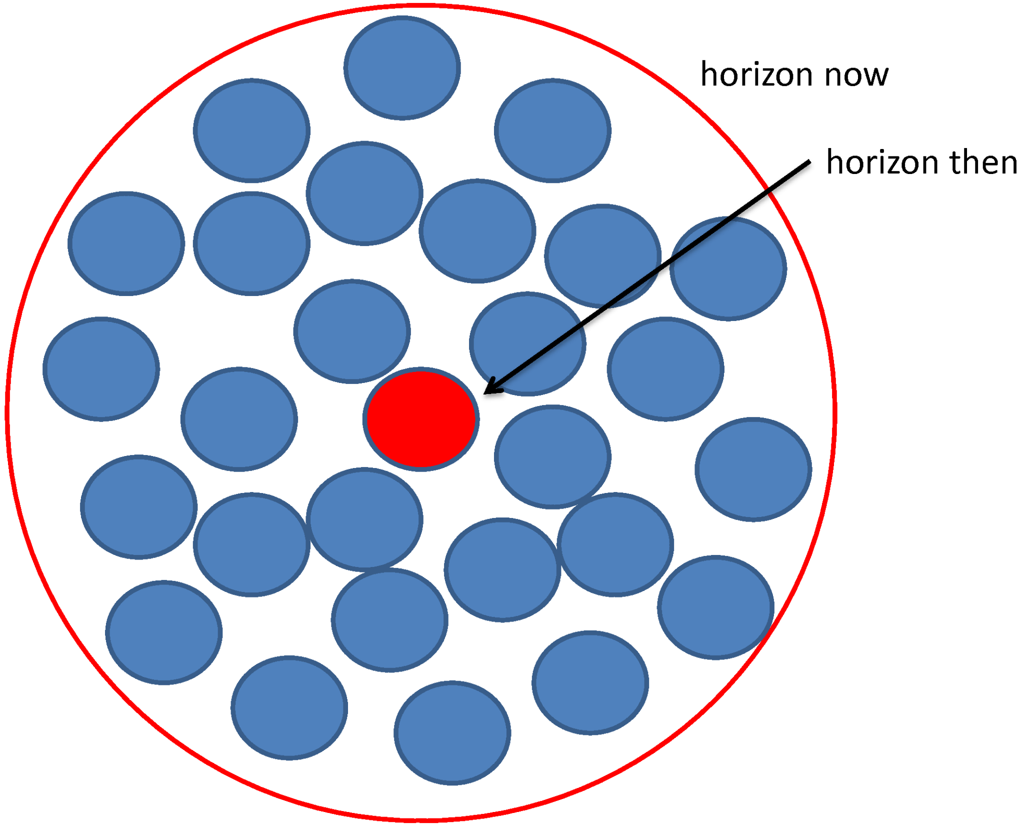
- The causal CMB volume at recombination time $t_{RC} = 380000 \ yrs$ (=when CMB photons escaped): $V \sim d_{RC}^3 = (2t_{RC})^3$.
- The causal CMB volume as seen today: $V_{CMB} \sim d_{\text{now}}^3 = (a_{\text{now}} d_{RC}/a_{RC})^3$

- The volume of the visible universe: $V_U \sim (t_{\rm now}/(1-\frac{2}{3}))^3$

Plug in $a_{\rm now}/a_{RC} = 1100$ to get the number of regions today that were not in causal contact at recombination

$$N = V_U/V_{CMB} \sim 8 \times 10^4 \;\; \text{check!} \tag{20}$$

Yet CMB has the same temperature everywhere. How come?


horizon now
horizon then

Solution again: de Sitter space

In de Sitter photon travels the coordinate distance

$$\int dx = \int dt a_0^{-1} e^{-H_0 t} = \frac{1}{H_0 a_0}\left(1 - e^{-H_0 t}\right)$$
$$\Rightarrow \quad d_\gamma = a(t)x = \frac{1}{H_0}\left(e^{H_0 t} - 1\right) \simeq \frac{1}{H_0} e^{H_0 t} \tag{21}$$

Two observers separated by an initial distance d_i will at late times find themselves at a distance $d(t) = d_i e^{H_0 t}$. They cannot communicate if $d(t) > d_\gamma(t)$ or if

$$d_i > \frac{1}{H_0} \ . \tag{22}$$

Thus de Sitter space has an event horizon.

Solution cont ...

Assume de Sitter expansion stops at $a_{\rm end}$ and the vacuum energy ρ_Λ is converted to radiation by

$$\rho_\Lambda = \frac{3H_0^2}{8\pi G} \equiv M_P^2 \Lambda^2 \sim T_0^4 \ . \tag{23}$$

(Λ is the cosmological constant). This would then mean the beginning of the "normal" Big Bang expansion with $a \sim t^{1/2}$.

- Two points A and B initially in causal contact $(d_A B(t_0) < H_0^{-1})$ will at later times lose causal contact because of the exponential expansion: $d_A B(t) > H_0^{-1}$. A is said to leave B's horizon.
- When exponential expansion ends and "normal" hot Big Bang expansion begins, the horizon grows as $d_H \sim t$ while $d_A B(t) \sim t^p$.
- Eventually $d_H > d_A B(t)$ and A and B are again in causal contact. A is said to re-enter B's horizon.

Solution cont ...

We should require that every point in the CMB sky as seen today has initially been in causal contact; i.e. the size of the visible universe at the end of inflation is $< H_0^{-1}$. It has then been stretched by inflation as

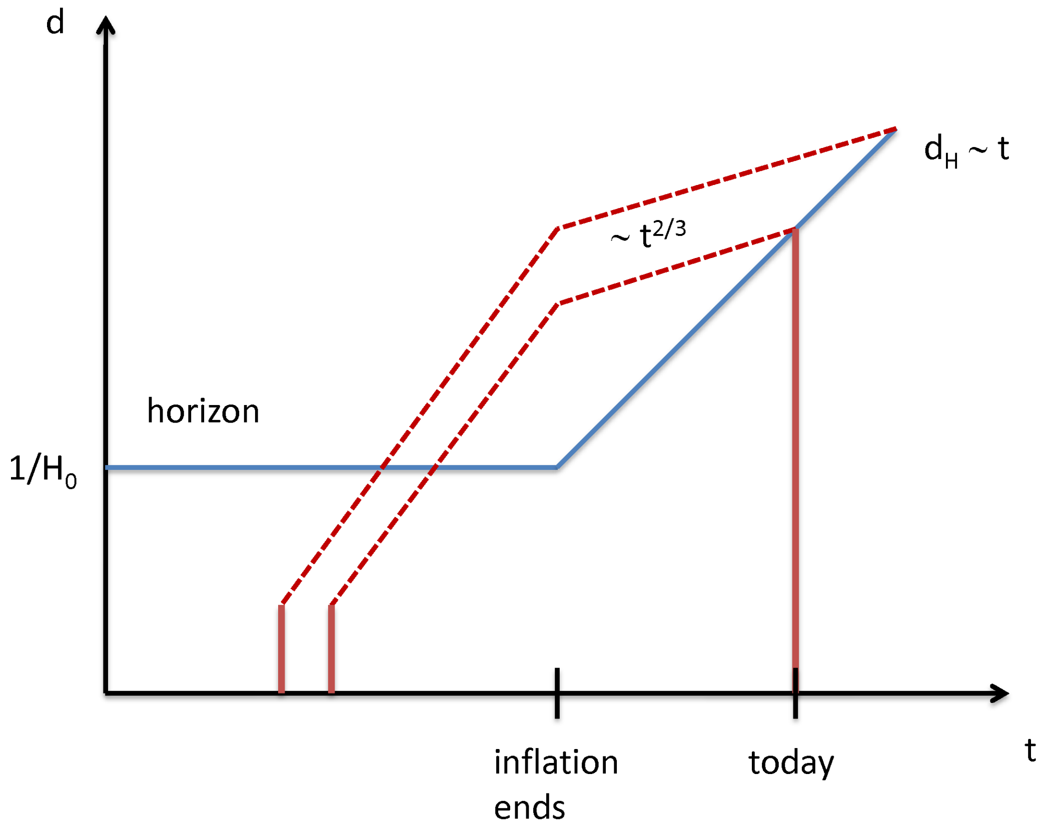
$$d_U = \frac{a_{\rm now}}{a_{\rm end}} e^{H_0 \tau} H_0^{-1} = \frac{T_0}{T_{\rm now}} e^{N} H_0^{-1} \tag{24}$$

where τ is the duration of inflation and N is the number of e-folds (and never mind the detailed expansion history now). The actual observed size of the universe today is very roughly $d_{\rm now} \sim 3/2H_{\rm now}$ (this does not account for dark energy) so that requiring $d_U > d_{\rm now}$ translates into

$$N > \ln \left[\frac{d_{\rm now T_0}}{d_U T_{\rm now}} \right] \sim 60 \tag{25}$$

for reference values $T_0 \sim 10^{15}$ GeV, $T_0 \simeq 3$K, $H_{\rm now} \simeq 70 kms^{-1}/Mpc$. (Check!)

Problem solved if inflation lasted more than 60 e-folds!



d
$d_H \sim t$
$\sim t^{2/3}$
horizon
$1/H_0$
inflation
ends
today
t

If $\rho_\Lambda = const$, then

$$\dot{\rho} + 3H(p+\rho) = 0 \Rightarrow p = -\rho \ . \tag{26}$$

Can we do this with a scalar field?

$$S = \int d^4x \sqrt{-g}\mathcal{L} \tag{27}$$

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \\ &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{a^2}(\nabla\phi)^2 - V(\phi)\ . \end{aligned} \tag{28}$$

Energy-momentum tensor:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial^\mu \phi - g^{\mu\nu}\mathcal{L} \tag{29}$$

$$\begin{aligned}\rho &= T_{00} = \dot{\phi}^2 - \left[\frac{1}{2}\left(\dot{\phi}^2 - \frac{1}{a^2}(\nabla\phi)^2\right) - V\right] \\ &= \frac{1}{2}\dot{\phi}^2 + \frac{1}{a^2}(\nabla\phi)^2 + V \\ T_{kk} &= (\partial_k\phi)^2 + a^2\left[\frac{1}{2}\left(\dot{\phi}^2 - \frac{1}{a^2}(\nabla\phi)^2\right) - V\right]\ . \end{aligned} \tag{30}$$

Let us define pressure p in comoving coordinates as

$$p = T_{kk}/a^2 = \frac{1}{2}\dot{\phi}^2 - \frac{1}{6}\frac{1}{a^2}(\nabla\phi)^2 - V \tag{31}$$

where the factor 1/6 comes about because isotropy is assumed here.

Note that wrinkles in the field tend to even out: $\nabla\phi$ comes always with a factor $1/a$. Thus

$$\frac{p}{\rho} \simeq \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V} \simeq -1 \tag{32}$$

if $\dot{\phi}^2 \ll V$.

Need a slowly moving field.

Equation of motion

Euler-Lagrange:

$$
\begin{aligned}
0 &= \frac{\partial\sqrt{-g}\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\sqrt{-g}\mathcal{L}}{\partial\partial_\mu\phi} \\
&= -a^3\frac{\partial V}{\partial\phi} - \partial_\mu \frac{\partial a^3 g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}{\partial\partial_\mu\phi} \\
&= -a^3 V' - \partial_0(a^3\partial_0\phi) + a\partial_k(\partial_k\phi) \\
\Rightarrow &\quad \ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + V' = 0 \ . \qquad (33)
\end{aligned}
$$

If we require $\dot{\phi}^2 \ll V$, then we should also require $\dot{\phi}\ddot{\phi} \ll \dot{V} = V'\dot{\phi}$ or $\ddot{\phi} \ll V'$.

This is called **slow roll**.

Free field

$$V = \frac{1}{2} m^2 \phi^2 \tag{34}$$

Let $\phi = A(t) e^{i\mathbf{k}\cdot\mathbf{x}}$. Then

$$\Rightarrow \left(\partial_0^2 + 3H\partial_0 + \frac{k^2}{a^2} + m^2 \right) A(t) = 0 \ . \tag{35}$$

Eqm has 1) friction term; 2) modes redshift for $k \neq 0 \rightarrow$ it makes sense to talk about comoving momentum $\tilde{\mathbf{k}} = \mathbf{k}/a$.

- typically at $t = 0$, $A = A_0$, $\dot{A} \simeq 0 \Rightarrow$ motion begins as harmonic oscillation until

- $H\dot{A} \gg (\tilde{k}^2 + m^2)A$

$$\frac{\ddot{A}}{\dot{A}} \simeq -3\frac{\dot{a}}{a} \Rightarrow \dot{A} = Ca^{-3} \ . \tag{36}$$

  For IR modes, A remains fixed until $H \simeq m$.

Thus typically

- gradients die away

- zero mode oscillates

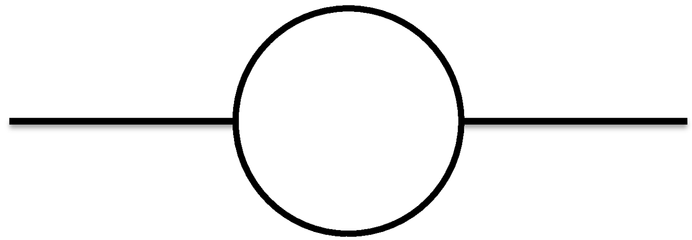
Homogeneous field

Eqm:

$$\ddot{\phi}(t) + 3H\dot{\phi}(t) + \Gamma\dot{\phi}(t) + V'(\phi(t)) = 0 \tag{37}$$

where we have added by hand the decay width Γ. Cannot be derived from the Lagrangian as $\Gamma = \mathrm{Im}E$. (In field theory can be computed as a loop contribution to self-energy - so the field is not really free.)

Exercise: For a free field in flat space, $\phi_k(t) = e^{-\Gamma t/2}\phi_k(0)\cos(E_k t)$.

$\Gamma = \mathrm{Im}$ 

For homogeneous field:

- decay can begin only when $\Gamma \gtrsim H$
- meanwhile, the amplitude decays
- if ϕ is the only energy component, then

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2, \;\; p = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2 \tag{38}$$

and $H^2 = 8\pi G\rho/3$. Integrated over one oscillation cycle one finds

$$\langle p\rangle \propto \int_0^{\omega t=2\pi} dt \frac{m}{2}\left(\sin^2(\omega t) - \cos^2(\omega t)\right) = 0 \tag{39}$$

Hence, the mean equation of state parameter for an oscillating (free) scalar field is $w = 0 \Rightarrow$ oscillating field behaves as cold matter with $\rho \propto a^{-3}$ and $a \propto t^{2/3}$.

Slow roll inflation

Assume

$$V = V_0 - a\phi - b\phi^2 - \dots \tag{40}$$

when $\phi \simeq 0$. Very flat potential $\rightarrow H^2 \simeq 8\pi G V_0/3 \simeq const. \equiv H_0^2$.

Assume initially $\phi \simeq 0$, $\dot{\phi} \simeq 0$

Slow motion (small acceleration) if $\ddot{\phi} \ll |V'|$, whence the slow roll equation of motion is

$$3H_0\dot{\phi} \simeq -V' \tag{41}$$

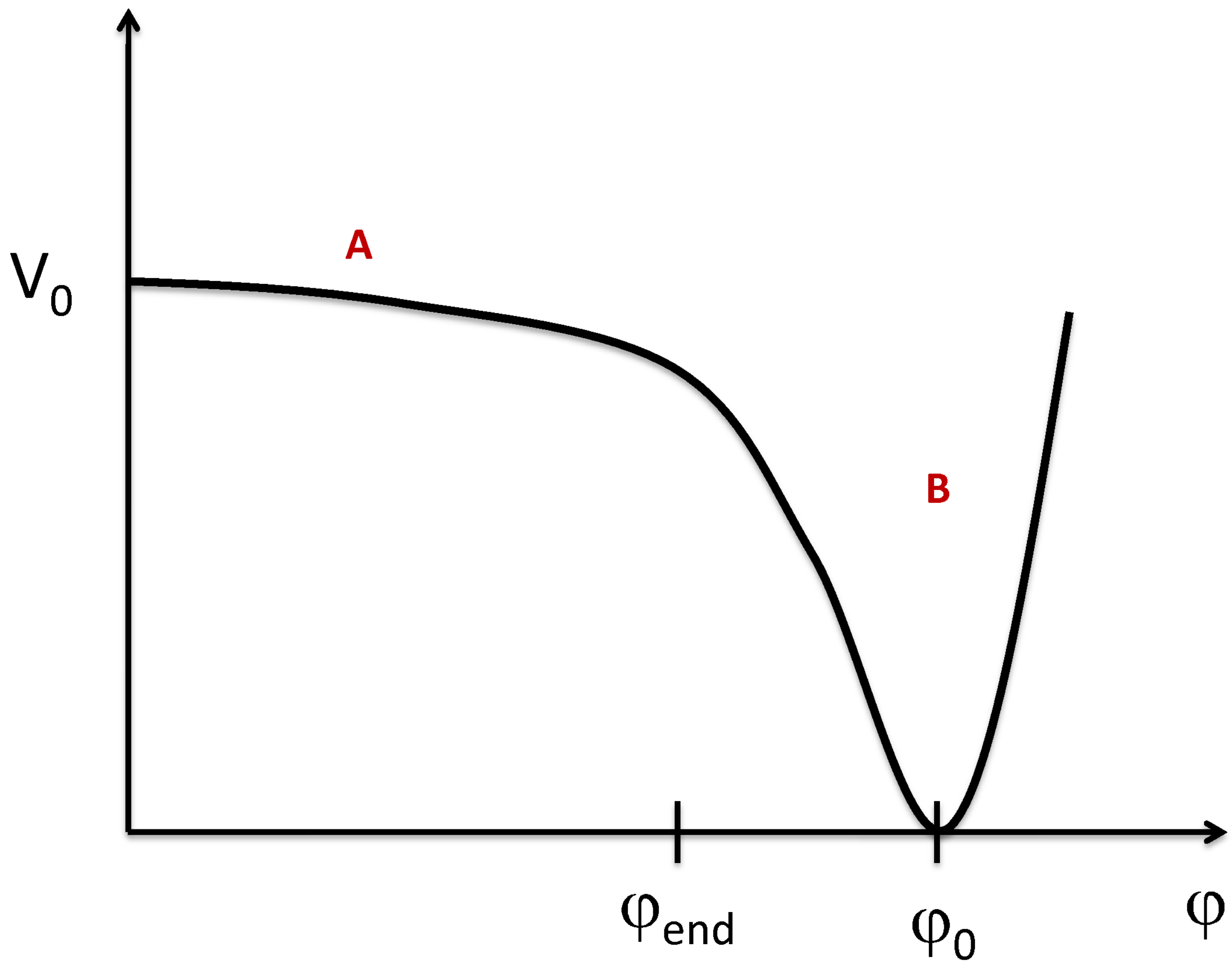


Figure 1: The inflaton potential

Slow roll parameters

Since $V \gg \dot{\phi}^2$, from the slow roll eqm

$$1 \gg \frac{1}{V}\left(\frac{V'}{3H}\right)^2 \simeq \frac{1}{V_0}\frac{V'}{3 \times 8\pi G V_0} \tag{42}$$

$\Rightarrow$ define

$$\epsilon \equiv \frac{M_P^2}{16\pi}\left(\frac{V'}{V}\right)^2 \equiv \frac{M}{2}\left(\frac{V'}{V}\right)^2 \ll 1 \tag{43}$$

where $M^2 = M_P^2/(8\pi)$ defines the reduced Planck mass. Taking a derivative

$$\frac{d}{d\phi}(V')^2 = 2V'V'' \ll \frac{16\pi}{M_P^2} 2V'V \Rightarrow \frac{V''}{V} \ll \frac{16\pi}{M_P^2} \tag{44}$$

$\Rightarrow$ define

$$\eta \equiv \frac{M_P^2}{8\pi}\frac{V''}{V} \equiv M^2 \frac{V''}{V} \ll 1 \tag{45}$$

Inflation ends when $\epsilon, \eta \simeq 1$.

Number of e-folds

$$
\begin{aligned}
N &\equiv \ln \frac{a(t_{\rm end})}{a(t_0)} = \int_{t_0}^{t_{\rm end}} dt H(t) \\
&= \int_{\phi_0}^{\phi_{\rm end}} d\phi \frac{H}{\dot{\phi}} = -\int_{\phi_0}^{\phi_{\rm end}} d\phi \frac{3H^2}{V'} \\
&= -\frac{1}{M^2} \int_{\phi_0}^{\phi_{\rm end}} d\phi \frac{V}{V'} \qquad (46)
\end{aligned}
$$

where $t_{\rm end}$ corresponds to $\epsilon(\phi) \simeq 1$.

Example: Let

$$V = V_0 - \mu^3\phi - \frac{1}{2}x\mu^2\phi^2, \quad \mu^4 \ll V_0 \tag{47}$$

Show that with $V_0^{1/4} \simeq 10^{-5}M$, $x \simeq 10^{-1}$, $\mu \simeq 10^{-10}M$ one finds $N \simeq 460$.

Reheating

Obviously, inflation must also end.

In region B, we write

$$V \simeq \frac{1}{2} m_\phi^2 (\phi - \phi_0)^2 + \cdots \equiv \frac{1}{2} m_\phi^2 \xi^2 + \ldots \tag{48}$$

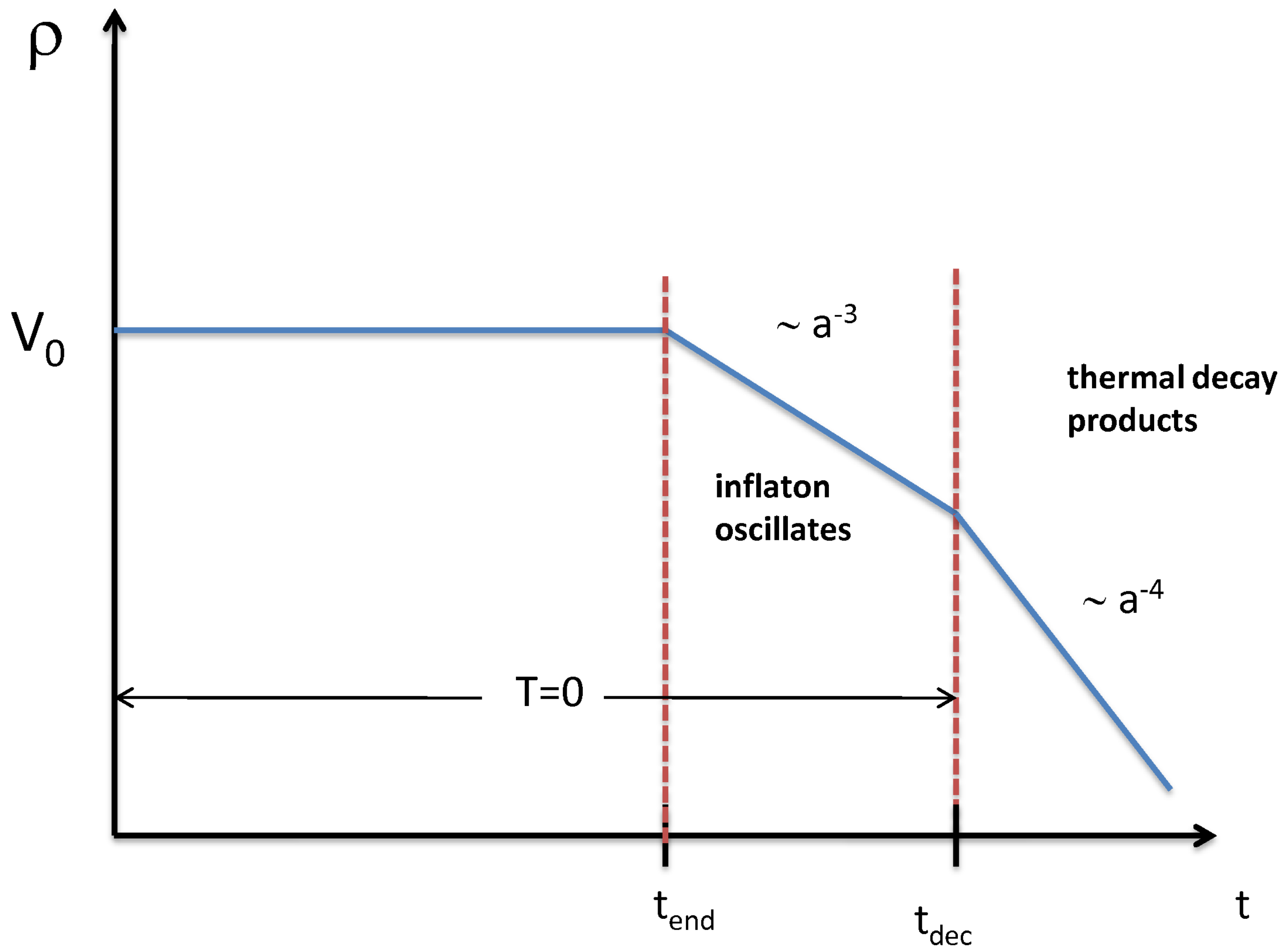
where m_ϕ is the physical inflaton mass and

$$\begin{aligned} \xi &= \xi_0 \left(\frac{a_0}{a(t)} \right)^{3/2} \cos m_\phi t \ , \\ \rho_\xi &= m_\phi^2 \xi_0^2 \left(\frac{a_0}{a(t)} \right)^3 \end{aligned} \tag{49}$$

until $\Gamma > H$. In the sudden decay approximation energy density is transformed into radiation instantenously so that

$$\rho_{\rm thermal} = \frac{g_* \pi^2}{30} T_{\rm RH}^4 = m_\phi^2 \xi_0^2 \left(\frac{a_0}{a(t_{\rm dec})} \right)^3 \tag{50}$$

with $t_{\rm dec} = \Gamma^{-1} = H({\rm dec})$.



ρ
V0
~ a-3
thermal decay
products
inflaton
oscillates
~ a-4
T=0
tend
tdec
t

Figure 2:

Parametric resonance/preheating

Simple inflaton decay often not efficient enough; can be enhanced by parametric resonance.

Let χ be a field that couples to the inflaton with

$$V = V(\phi) + \frac{1}{2} m_\chi \chi^2 - g^2 \phi^2 \chi^2 \tag{51}$$

Here $\phi - \phi_0 \to \phi$.

- For large ϕ the mass of χ is $M_\chi = g\phi \gg m_\phi \Rightarrow$ inflaton cannot decay into χ.
- At the end of inflation, the inflaton oscillates $\Rightarrow$ periods with $g\phi \simeq 0$ when decay is possible

$\Rightarrow$ bursts of particle production.

Eqm for χ in the evolving background of ϕ is

$$\begin{aligned} 0 &= \ddot{\chi} + 3H\dot{\chi} + (m_\chi^2 + g^2\phi^2)\chi \\ &\Rightarrow \ddot{\chi}_k + 3H\dot{\chi}_k + (\frac{k^2}{a^2} + m_\chi^2 + g^2\Phi^2(t)\sin^2(m_\phi t))\chi_k = 0 \\ &\Leftrightarrow \chi_k'' + (A - 2q\cos(2z))\,\chi_k = 0 \end{aligned} \tag{52}$$

with $z = m_\phi t$ (this is the Mathieu equation), where we now assume $m_\chi \simeq 0$, and

$$A = \frac{k^2}{a^2} + 2q, \quad q = \frac{g^2\Phi^2(t)}{4m_\phi^2} \tag{53}$$

- Mathieu eq has instability bands in the (A, q) plane

Mathieu eq instability bands

Lev Kofman, Andrei Linde, Alexei Starobinsky: hep-th/9405187

HEP-TH 9405187

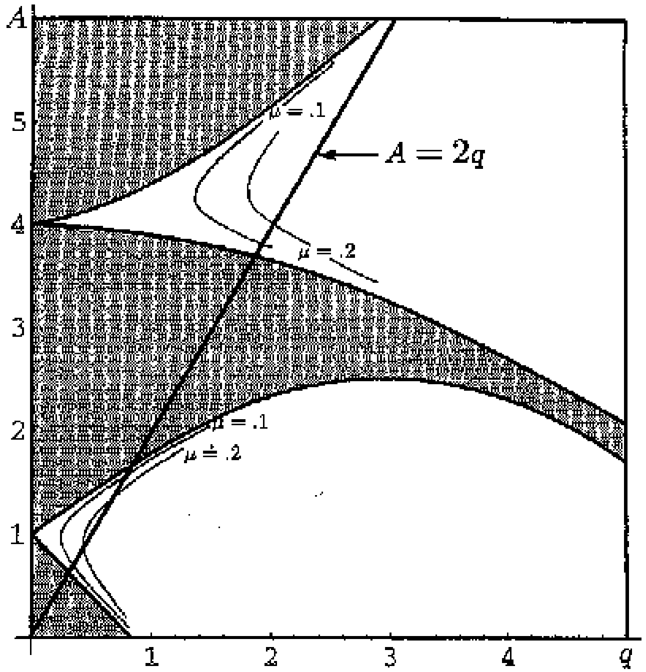
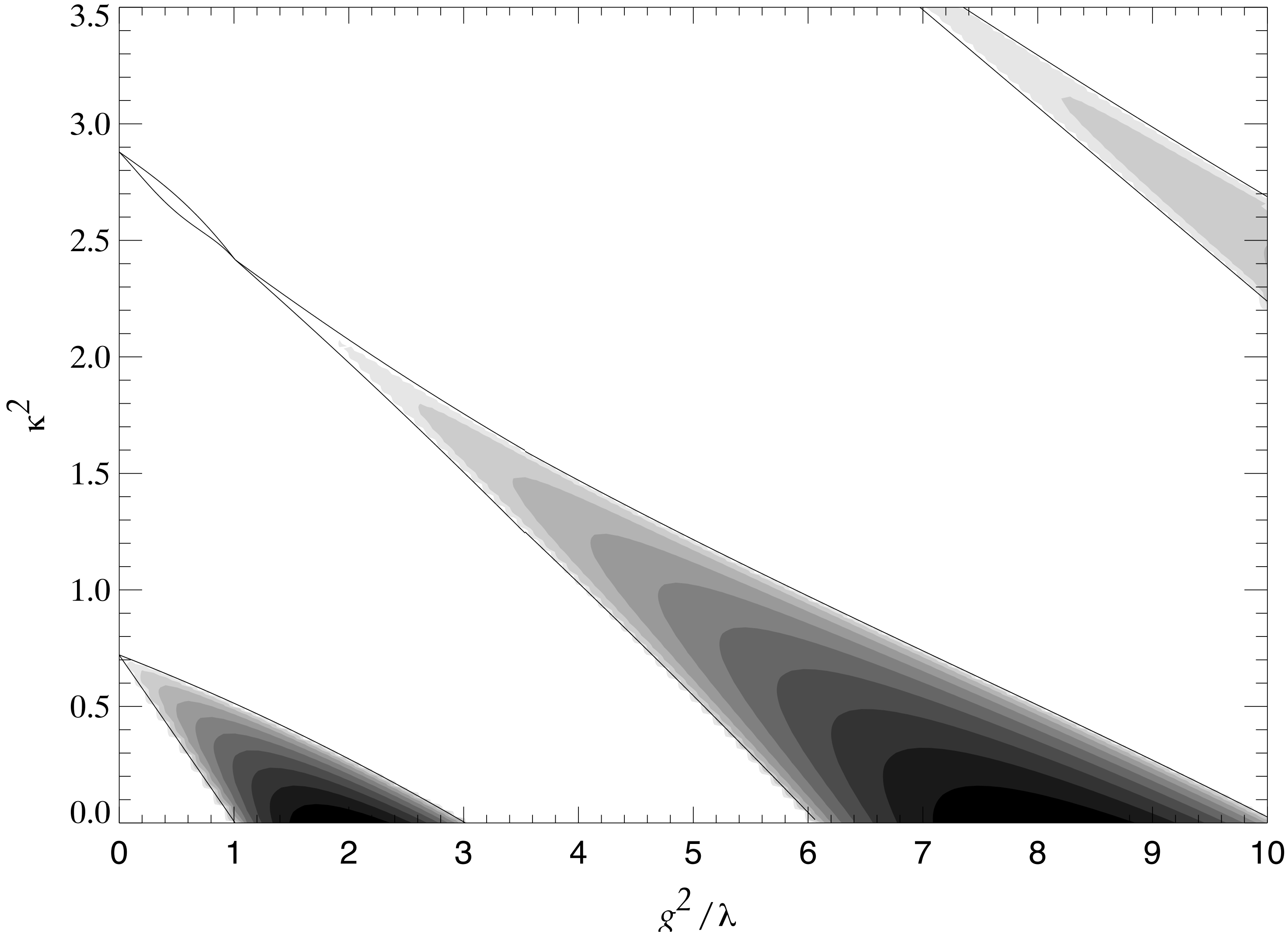
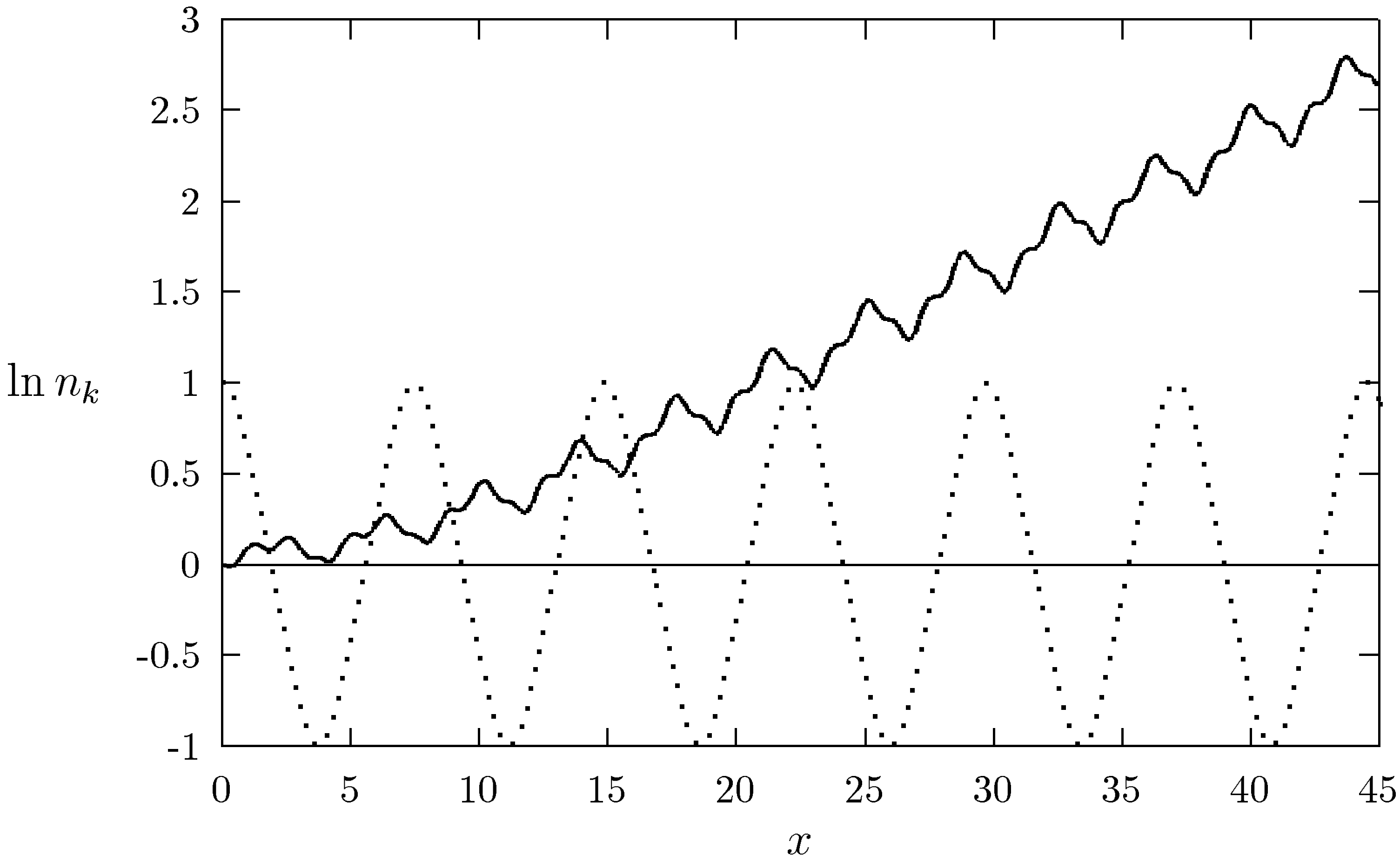


Fig. 1 The sketch of the stability/instability chart of the canonical Mathieu equation (5). Parameters A and q are defined in the text after eq. (5). White regions correspond to the instability (resonance) bands, grey regions correspond to the stability bands. Some contours of the parameter μ_k in the first and second instability bands are shown. The straight line crossing the chart corresponds to $A = 2q$.

Instability bands, $\lambda\phi^4$



Growth of particle number



Questions:

- how does the universe get populated by SM degrees of freedom?
- what exactly is ϕ - must it be singlet field?
- can the Higgs be the inflaton?
- scalars of MSSM?

Inflatory perturbations

The main reason for believing in inflation

Consider

$$\phi(x,t) = \phi_0(t) + \delta\phi(x,t) \tag{54}$$

- ϕ_0 is the slow rolling background field
- $\delta\phi(x,t)$ is the small perturbation
- inflationary perturbations $\Rightarrow$ density perturbations

$$\delta\rho \propto \delta V(\phi(x,t)) = V'(\phi_0)\delta\phi(x,t) \tag{55}$$

To linear order

$$\begin{aligned} V'(\phi) &= V'(\phi_0(t) + \delta\phi(x,t)) = V'(\phi_0) + V''(\phi_0)\delta\phi(x,t) + \ldots \\ 0 &= \delta\ddot{\phi} + 3H(\phi_0)\delta\dot{\phi} - \nabla^2\delta\phi + m_\phi^2\delta\phi \\ &\Leftrightarrow \delta\ddot{\phi}_k + 3H(\phi_0)\delta\dot{\phi}_k + \left(\frac{k^2}{a^2} + m_\phi^2\right)\delta\phi_k = 0 \end{aligned} \tag{56}$$

- Obviously, for massless fields and for long-wavelenght fluctuations with $k \to 0$, one finds that $\delta\phi_k \to const \Rightarrow$ once beyond the horizon, the perturbation freezes

- slow roll condition requires $m_\phi \ll H$
- for a small perturbation well within the horizon, $k/a \gg H$

Hence at early times we may write

$$\delta\ddot{\phi}_k + 3H(\phi_0)\delta\dot{\phi}_k + \frac{k^2}{a^2}\delta\phi_k = 0 \tag{57}$$

What is the source of perturbation? Note that the above also holds for the vacuum fluctuation

$$\delta\phi_k = w_k \tag{58}$$

At small distance scales curvature can be neglected $\Rightarrow$ consider initial flat space quantum fluctuations

Quantum fluctuations of a scalar field

- quantize free field in Minkowski space:

$$\begin{aligned} \phi_k &= w_k(t)a_k + w_k^*(t)a_{-k}^\dagger \\ w_k &= V^{-1/2}\sqrt{\frac{1}{2E_k}}\exp(-iE_kt) \end{aligned} \tag{59}$$

with

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}, \; [a_k, a_{k'}] = 0 \; . \tag{60}$$

- define Minkowski vacuum

$$a_k|0\rangle = 0, \; a_k^\dagger|0\rangle \propto |k\rangle \tag{61}$$

In the vacuum

- VEV is zero

$$\langle 0|\phi_k|0\rangle = \sum_k \langle 0|\phi_k|0\rangle e^{ikx} = \sum_k \langle 0|w_k(t)a_k + w_k^*(t)a_{-k}^\dagger|0\rangle e^{ikx} = 0 \tag{62}$$

- variance is not

$$\begin{aligned}\langle 0|\phi^2|0\rangle &= \sum_{kk'} e^{ikx+ik'x}\langle 0|w_k a_k w_k' a_k' + w_k^* a_{-k}^\dagger w_k'^* a_{-k'}^\dagger \\ &+ w_k a_k w_k'^* a_{-k'}^\dagger + w_k' a_k' w_k^* a_{-k}^\dagger|0\rangle \\ &= \sum_k |w_k|^2 \end{aligned} \tag{63}$$

Assume: the initial perturbation $\delta\phi_k$ at $t \to -\infty$ should be identified with the root-mean-square quantum fluctuation.

Let us define the power spectrum $P(k)$ of field fluctuations:

$$\begin{aligned} P(k) &= \frac{V}{(2\pi)^3} k^2 d\Omega \langle |\delta_k|^2 \rangle \\ &= \frac{V k^3}{4\pi^2 E_k} \langle |\delta_k|^2 \rangle \end{aligned} \tag{64}$$

Thus: find the solution of

$$\ddot{w}_k + 3H(\phi_0)\dot{w}_k + \frac{k^2}{a^2} w_k = 0, \tag{65}$$

with $w_k(-\infty)$ given by (59) in a box of size L; let us also ignore the variation of H: $H(\phi_0(t)) \simeq H_0$.

- the result:

$$w_k = \frac{1}{L^{3/2}} \frac{H}{(2k^3)^{1/2}} \left(i + \frac{k}{aH} \right) \exp\left(\frac{ik}{aH} \right) \quad . \tag{66}$$

  (exercise: check).

What is the amplitude at horizon crossing?

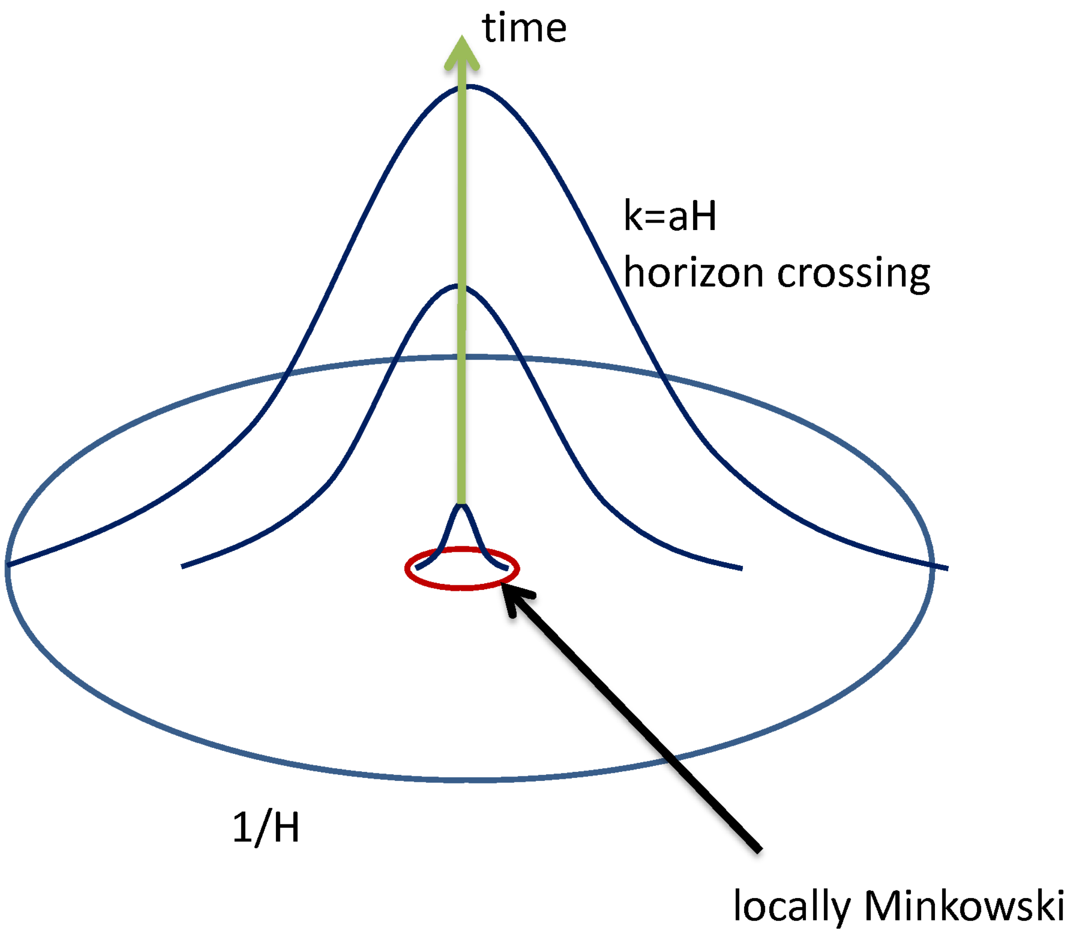


Figure 3: perturbation exits the horizon

Check first that $t \to -\infty$ is ok: at early times $a \simeq 0$ so that

$$
\begin{aligned}
\frac{k}{aH} &= \frac{k}{aH}|_{t=t_0} + \frac{d}{dt}\frac{k}{aH}|_{t=t_0}(t-t_0) + \dots \\
&= \frac{k}{aH}|_{t=t_0} - \frac{\dot{a}}{a^2}\frac{a}{\dot{a}}k|_{t=t_0}(t-t_0) + \dots = const - \frac{k}{a}|_{t=t_0}(t-t_0) + \dots \\
\Rightarrow &\quad w_k(t \to -\infty) = \frac{1}{L^{3/2}}\frac{1}{(2k^3)^{1/2}}\frac{k}{a}e^{i\times const}e^{-ikt/a}
\end{aligned}
\tag{67}
$$

and since $\delta\phi_k = w_k$ is massless, $E_k = k/a$ and

$$
\begin{aligned}
w_k(t \to -\infty) &= \frac{1}{(aL)^{3/2}}\frac{k/a}{(2(k/a)^3)^{1/2}}e^{i\times const}e^{-iE_k t} \\
&= \frac{1}{V_{\text{comoving}}^{3/2}}\frac{1}{\sqrt{2E_k}}e^{-iE_k t}
\end{aligned}
\tag{68}
$$

which is the flat space result (59).

And now, finally: the spectrum at horizon exit t_*: the perturbation amplitude is

$$\langle|\delta_k|^2\rangle = |w_k|^2 = \frac{H^2(t_*)}{2L^3k^3} \tag{69}$$

and thus at the horizon exit

$$P(k,t_*) = \frac{k^3}{4\pi^2 V E_k}\frac{H^2(t_*)}{2L^3k^3} = \left(\frac{H(t_*)}{2\pi}\right)^2 \equiv \left(\frac{H}{2\pi}\right)^2 |_{k=aH} \ . \tag{70}$$

- field perturbation $\Leftrightarrow$ density perturbation
- density pertubation $\Leftrightarrow$ metric perturbation

$$T_{\mu\nu} \rightarrow T_{\mu\nu} + \delta T_{\mu\nu} \Leftrightarrow g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu} \tag{71}$$

In so-called Newtonian gauge

$$ds^2 = (1 + 2\Phi)dt^2 - a^2(t)(1 + 2\Psi)dx^2 \tag{72}$$

where $\Psi,\ \Phi$ are called the Bardeen potentials

- perfect non-viscous fluid: $\Psi = \Phi$
- above ignores vector and tensor perturbations (=gravitational waves)

But what is physical?

- In GR one is free to change coordinates: $g_{\mu\nu} \longrightarrow g'_{\mu\nu}$
- One e.g. could choose $g_{\mu\nu}$ such that $\delta\rho(x,t) = 0$ (”constant density hypersurface”)$\Leftrightarrow$ choosing a gauge

$\Rightarrow$ need gauge invariant description of the perturbation

One can show that the comoving curvature perturbation

$$\mathcal{R} = -\frac{H}{\dot{\phi}}\delta\phi \tag{73}$$

is

- gauge invariant
- remains constant outside horizon
- Bardeen potential $\Psi = \frac{3}{5}\mathcal{R}$

Outside horizon $\mathcal{R} = -\zeta$, the curvature perturbation.

How to relate theory to observations?

- CMB observations yield correlators such as

$$\langle \frac{\delta T}{T} \frac{\delta T}{T} \rangle \equiv P_T(k) \tag{74}$$

  or the CMB temperature power spectrum.

- For large angular distances

$$\frac{\delta T}{T} = -\frac{1}{3}\Psi \tag{75}$$

  This is the **Sachs-Wolfe effect**.

Thus, the task is to compute $\langle \mathcal{R}_k \mathcal{R}_k \rangle$

And now: the prediction

$$
\begin{aligned}
P_{\mathcal{R}}(k) &= \langle \mathcal{R}_k \mathcal{R}_k \rangle \\
&= \frac{H^2}{\dot{\phi}^2} \langle \delta\phi_k \delta\phi_k \rangle \\
&= \frac{H^2}{\dot{\phi}^2} \left(\frac{H}{2\pi} \right)^2 |_{k=aH} \qquad (76)
\end{aligned}
$$

Using the slow-roll condition $\dot{\phi} = -V'/3H$

$$
\begin{aligned}
P_{\mathcal{R}}(k) &= \frac{1}{4\pi^2} \frac{9}{(V')^2} \frac{V}{3M^2} \\
&= \frac{1}{24\pi^2} \frac{V}{M^4} \frac{1}{\epsilon} \qquad (77)
\end{aligned}
$$

Thus the amplitude of the fluctuation (=Sachs-Wolfe) is

$$\begin{aligned}\delta_H^2 &= \frac{2}{5}P_{\mathcal{R}}(k) = \frac{1}{150\pi^2}\frac{V}{\epsilon}\frac{1}{M^4} = (1.9\times 10^{-5})^2 \text{ (COBE)} \\ \Rightarrow & \left(\frac{V}{\epsilon}\right)^{1/4} = 0.027M \end{aligned} \tag{78}$$

Given the primordial spectrum, can predict the whole CMB TT-spectrum (see Kurki-Suonio)

Spectral index

Phenomenologically one writes

$$P_{\mathcal{R}} = Ak^{n-1} \ , \tag{79}$$

where n is the spectral index (measured from CMB). This can now be predicted:

$$n - 1 = \frac{d \ln P_{\mathcal{R}}}{d \ln k} \tag{80}$$

Since $k = aH$,

$$d \ln k = \frac{da}{a} + dHH \simeq H dt \ . \tag{81}$$

Slow roll:

$$\dot{\phi} + 3HV' = 0 \to dt = -3H d\phi/V' \tag{82}$$

Thus

$$\frac{d}{d \ln k} = -\frac{V'}{3H^2}\frac{d}{d\phi} = -M^2\frac{V'}{V}\frac{d}{d\phi} \tag{83}$$

$$
\begin{aligned}
\frac{d\epsilon}{d\ln k} &= -M^2\frac{V'}{V}\frac{1}{2}M^2\frac{d}{d\phi}\left(\frac{V'}{V}\right)^2 \\
&= -\frac{1}{2}M^4\frac{V'}{V}\left[-\left(\frac{V'}{V}\right)^3\times 2+\frac{2V'V''}{V^2}\right] \\
&= 4\epsilon^2-2\epsilon\eta \qquad (84)
\end{aligned}
$$

= the running of ϵ. Moreover:

$$
\frac{d\ln V}{d\ln k}=-M^2\frac{V'}{V}\frac{V'}{V}=-2\epsilon \qquad (85)
$$

Hence our prediction is

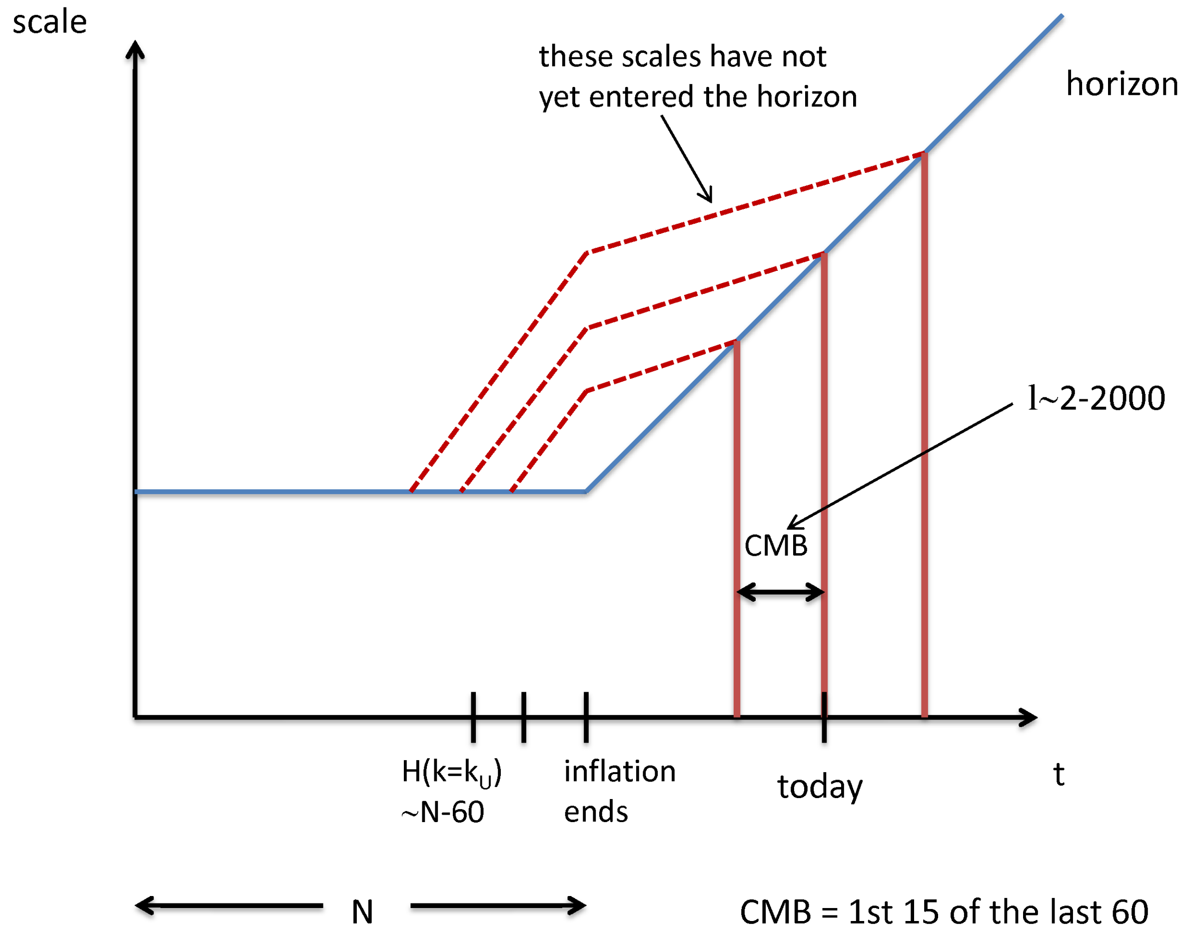
$$
\begin{aligned}
n-1 &= \frac{d\ln V}{d\ln k} - \frac{d\epsilon}{d\ln k} \\
&= -2\epsilon - 4\epsilon + 2\eta \\
&= 2\eta - 6\epsilon
\end{aligned}
\tag{86}
$$

WMAP7 (CMB alone): $1-n = 0.037 \pm 0.014$ Hence typically $\epsilon \sim \mathcal{O}(0.01)$ and

- typically $V^{1/4} \simeq 8.5 \times 10^{-3} M = 2.3 \times 10^{16}$ GeV
- during inflation $H \simeq 4 \times 10^{-5} M = 9.6 \times 10^{13}$ GeV

What exactly is observed?

What matters is $V(\phi)$ when the scales observed in CMB were generated


scale
these scales have not
yet entered the horizon
horizon
l~2-2000
CMB
H(k=k_U)
~N-60
inflation
ends
today
t
N
CMB = 1st 15 of the last 60

Example: chaotic inflation

$$V = \frac{1}{4}\lambda\phi^4 \tag{87}$$

Number of e-folds is

$$\begin{aligned} N &= -\int_{\phi_0}^{\phi_{\rm end}} d\phi \frac{V}{V'}\frac{1}{M^2} = -\frac{1}{4M^2}\int_{\phi_0}^{\phi_{\rm end}} d\phi\phi \\ &= \frac{1}{4M^2}\left(\phi_0^2 - \phi_{\rm end}^2\right) \simeq 64 \\ \Rightarrow &\quad \phi_0^2 \gg \phi_{\rm end}^2 \simeq \mathcal{O}\left(\text{few} \times M\right)^2 \end{aligned} \tag{88}$$

($\phi \gg M$ could be a problem). Thus

$$\begin{aligned} \epsilon &= \frac{1}{2}\left(\frac{V'}{V}\right)^2 = 8\frac{M^2}{\phi_0^2} = 0.031 \\ \eta &= M^2\frac{V''}{V} = 12\frac{M^2}{\phi_0^2} = 0.047 \end{aligned} \tag{89}$$

The observables are

$$\begin{aligned}
\delta_H^2 &= \frac{1}{150\pi^2}\frac{V}{\epsilon}\frac{1}{M^4} = 357\lambda = (1.9\times 10^{-5})^2 \\
&\Rightarrow \lambda \simeq 10^{-12} \\
n &= 1 + 2\eta - 6\epsilon = 0.908
\end{aligned} \tag{90}$$

Tensor-to-scalar ratio

In addition to scalar perturbations, there are also tensor perturbations $\delta g_{\mu\nu}$ = gravitational waves. These are

- massless fields $\Rightarrow$ subject to perturbations like scalars

$$"\phi" = \frac{M}{\sqrt{2}} \delta g^T_{\mu\nu} \tag{91}$$

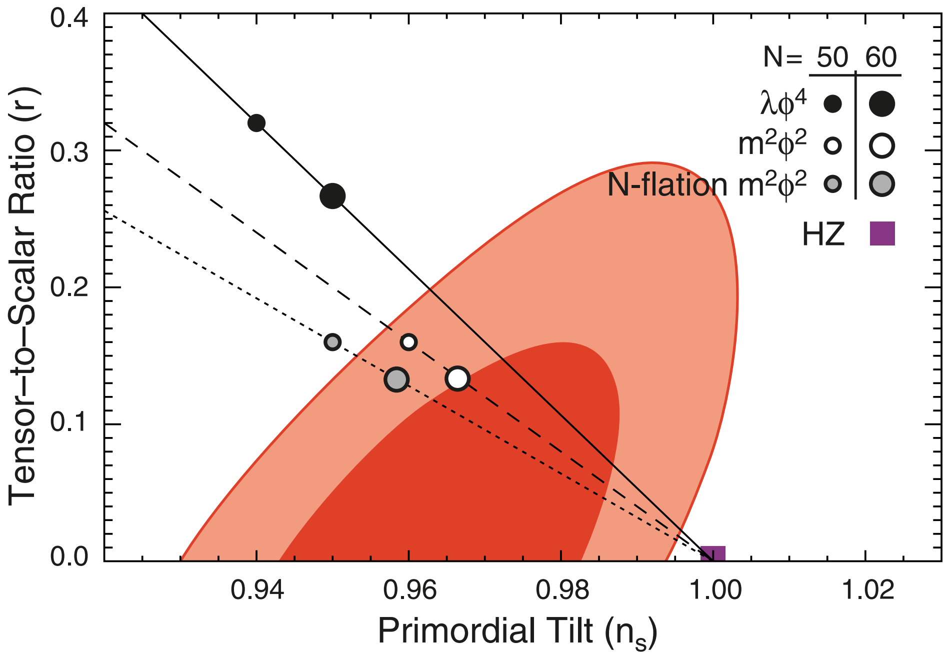
  Hence

$$P_{\rm grav} = \frac{2}{M^2} \left(\frac{H}{2\pi} \right)^2 |_{k=aH} \tag{92}$$

- In single field inflation one finds

$$\begin{aligned} \frac{P_{\rm grav}}{P_{\mathcal{R}}} &= \frac{2}{M^2} \left(\frac{H}{2\pi} \right)^2 \frac{1}{(H/\dot{\phi})^2} \frac{1}{(H/2\pi)^2} = \frac{6\dot{\phi}^2}{V} = \frac{6(V')^2}{3H^2V} \\ &= 6 \left(\frac{V'}{V} \right)^2 M^2 = 12\epsilon \end{aligned} \tag{93}$$

WMAP7 constraints on models



What is the inflaton?

- The Higgs? Low-scale inflation has a generic problem: we need $(V/\epsilon)^{1/4} \sim 0.03M$ so that with $V \ll M^4$, one must require $\epsilon \to 0$.

  - very flat potential: problem with radiative corrections: not flat enough

  - $n \to 1$ - ruled out?

  - possible way out: direct coupling to gravity through $\xi H^2 R \Rightarrow$ inflation at large Higgs field values (for recent discussion and references, see JHEP 0907:089,2009.)

- beyond SM: MSSM The scalar potential has flat directions with $V = 0$ up tp non-renormalizable terms. Once susy is broken, the potential reads (see Phys.Rept.380:99-234,2003)

$$V = \frac{1}{2}m^2\phi^2 + \frac{\phi^n}{M^{n-4}} \tag{94}$$

  which may be flat enough; the form depends on the direction, which are all classified and can be thought of as some combinations of squark and slepton fields (for most recent attempts and references, see 1004.3724 [hep-ph])

- usual assumption: inflaton is some singlet scalar

- Hybrid inflation

$$V = V(\phi, \sigma) \tag{95}$$

  where ϕ is the inflaton and σ is the "waterfall field": decoupling inflation and the end of inflation (can have faster reheating)

- models with supergravity
- models based on/inspired by strings
- multifield models: generically both adiabatic and isocurvature perturbations

Non-gaussianity

- initial field fluctuations uncorrelated $\Rightarrow$ gaussian fluctuations
- correlations = interactions $\Rightarrow$ non-gaussianity

  gravitational interactions, field interactions
- models with more than 1 field
- very topical

  probes directly the inflaton model
- observations: CMB is mostly gaussian - non-gaussianities must be small

$X, Y, \ldots$ are gaussian random variables:

- probability distribution

$$P(X) = N_X e^{-X^2/(2\sigma_X^2)} \tag{96}$$

- average is zero

$$\langle X \rangle = N_X \int dX X e^{-X^2/(2\sigma_X^2)} = 0 \tag{97}$$

- two-point correlators

$$\begin{aligned} \langle XY \rangle &= N_X N_Y \int dX dY X Y P(X) P(Y) \\ &= N_X N_Y \delta(X - Y) \int dX X^2 P(X) \equiv \sigma_X \delta(X - Y) \end{aligned} \tag{98}$$

- Gaussian fluctuations $\Rightarrow$ odd correlators vanish

$$\langle XYZ\rangle = 0 \ : \ \langle XY^2\rangle = 0; \ \langle X^3\rangle = 0 \tag{99}$$

- even correlators are related to the two-point correlator

$$\begin{aligned} \langle XYZU\rangle &= \delta(Z-U)\delta(X-Y)\langle X^2\rangle\langle Z^2\rangle + \ldots \\ &+ \delta(Z-U)\delta(X-Y)\delta(X-U)\langle X^4\rangle \end{aligned} \tag{100}$$

and

$$\langle X^4\rangle \propto \langle X^2\rangle^2 \tag{101}$$

$\Rightarrow$ the obvious place to look for non-gaussianity is the 3-point correlator, e.g. $\langle \delta T(x)\delta T(y)\delta T(z)\rangle$.

Let us write the curvature perturbation as

$$\zeta = \zeta_g - \frac{3}{5} f_{NL} \zeta_g^2 \tag{102}$$

where ζ_g is gaussian (and 3/5 a convention). This is called χ^2 non-gaussianity. We get a 3-point correlator

$$\begin{aligned} \langle \zeta\zeta\zeta \rangle &= \langle \zeta_g^3 \rangle - \frac{3}{5} f_{NL} \langle \zeta_g^4 \rangle \\ &\propto 0 + f_{NL} (\langle \zeta_g^2 \rangle)^2 \end{aligned} \tag{103}$$

The relation to observables is

$$\begin{aligned} \frac{\delta T}{T} &= -\frac{1}{3}\Phi = -\frac{1}{3}\left(\Phi_g + \tilde{f}_{NL}\Phi_g^2\right) \\ &= \tilde{f}_{NL} = -f_{NL} + \frac{5}{6} \end{aligned} \tag{104}$$

for the Sachs-Wolfe and likewise for the rest of CMB spectrum.

- WMAP limit is $-10 < f_{NL} < 74$ at 95% CL
- Given an inflaton potential, how to compute the 3-point-function?

Separate universe/ΔN formalism

- each point is considered as an independent FRW universe
- local scale factor is different at each point:

$$a^2(x,t) = a^2(t)e^{-2\psi(x,t)} \tag{105}$$

- $\Rightarrow$ expansion histories differ locally

$$N(x) = \int dt \frac{\dot{a}(x)}{a(x)} \tag{106}$$

- OK as long as gradients are small (OK as perturbations are known to be small)

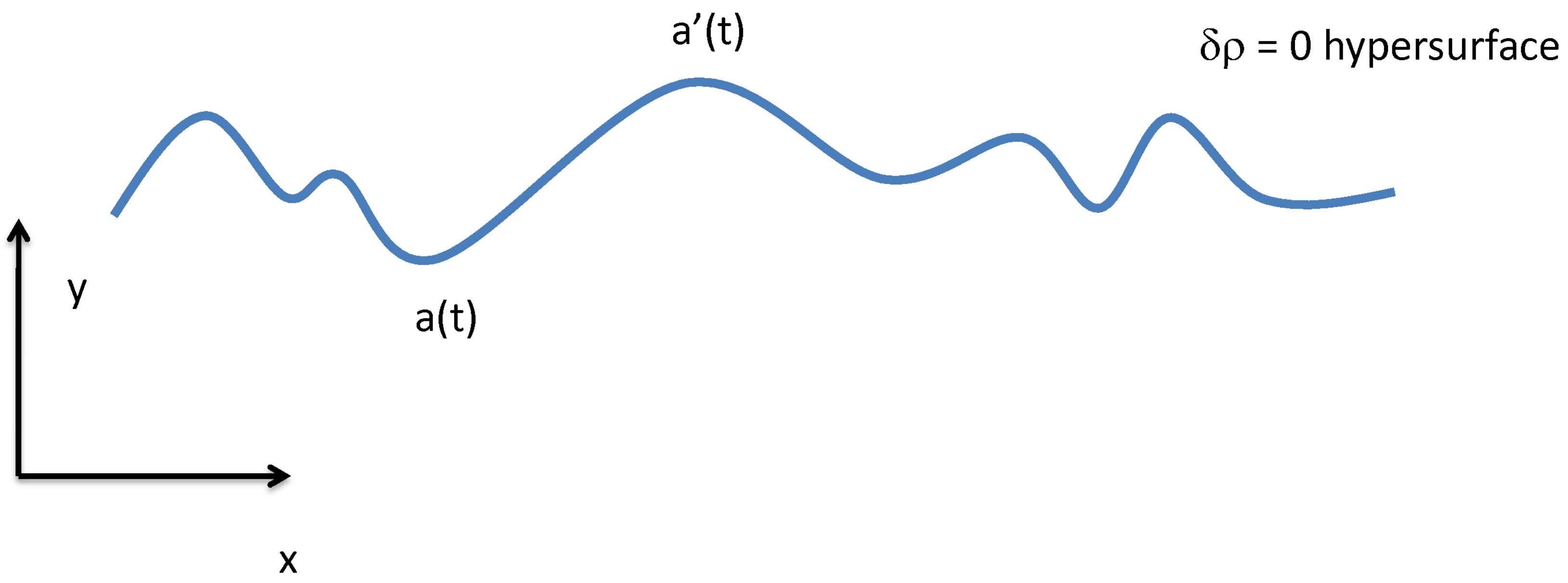


Figure 4: δN formalism

The number of e-folds felt by an observer at x and at time t is

$$N = \ln \frac{a(t)e^{-\psi(x,t)}}{a(t_i)e^{-\psi(x,t_i)}} = \psi(x,t_i) - \psi(x,t) + \ln \frac{a(t)}{a(t_i)} \tag{107}$$

Assume initially flat hypersurface with $\psi(x,t_i) = 0$. Then the curvature perturbation reads

$$\zeta = -\psi(x,t) = N - \ln \frac{a(t)}{a(t_i)} \equiv \Delta N \tag{108}$$

ΔN measures the shift in expansion between the flat and uniform density surfaces.

Recall that N is determined by the inflaton field: $N = N(\phi)$. Thus

$$\begin{aligned}
\zeta &= \Delta N = \int_{\phi_i}^{\phi} d\phi \frac{H}{\dot{\phi}} - \int_{\phi_i}^{\phi'} d\phi \frac{H}{\dot{\phi}} \\
&= \int_{\phi+\delta\phi}^{\phi} d\phi \frac{H}{\dot{\phi}}
\end{aligned} \tag{109}$$

$\Rightarrow$ we find the useful expansion

$$\zeta = N'(\phi)\delta\phi + \frac{1}{2}N''(\phi)\delta\phi^2 + \ldots \tag{110}$$

Back to non-gaussianity

Let's write

$$\begin{aligned} \zeta &\simeq N'(\phi)\delta\phi + \frac{1}{2}N''(\phi)\delta\phi^2 \\ &= N'(\phi)\delta\phi + \frac{N''}{2N'(\phi)^2}(N'(\phi)\delta\phi)^2 \\ &= \zeta_g + \frac{N''}{2N'(\phi)^2}\zeta_g^2 \end{aligned} \tag{111}$$

so that we find the simple relation

$$f_{NL} \simeq \frac{5}{6}\frac{N''}{N'(\phi)^2} \tag{112}$$

Recall

$$
\begin{aligned}
N' &= -\frac{H}{\dot{\phi}} \\
H^2 &= \frac{V}{3M^2},\;\; 3H\dot{\phi} = -V' \\
&\Rightarrow \;\; N' = \frac{V}{M^2 V'}
\end{aligned} \tag{113}
$$

$\Rightarrow$

$$N'' = \frac{1}{M^2} + \frac{VV''}{M^2 (V')^2} \tag{114}$$

$\Rightarrow$

$$\frac{N''}{N'^2} = \frac{M^4 V'^2}{V^2 M^2} - \frac{VV'' M^4 (V')^2}{M^2 (V')^2 V^2} = 2\epsilon - \eta \tag{115}$$

$\Rightarrow$

$$f_{NL} = -\frac{5}{12}(2\eta - 4\epsilon) \tag{116}$$

A correct calculation (Maldacena) yields $f_{NL} = -\frac{5}{12}(2\eta - 6\epsilon)$.

- single field inflation predicts $f_{NL} \sim \mathcal{O}(\epsilon)$
- generic multifield inflation models predict $f_{NL} \sim \mathcal{O}(1)$ (with exceptions)
- Planck is expected to be able to detect $|f_{NL}| \geq 5$
- detection: single field inflation ruled out, many models in big trouble
- non-gaussianity may also be generated after inflation
- preheating may lead to very large non-gaussianity (see e.g. JCAP 0808:002,2008.)

- 2-point correlator $\Rightarrow$ spectrum $P(k)$
- 3-point correlator $\Rightarrow$ bispectrum $P(k_1, k_2)$

  - correlator = triangles of different size and form

  - in general $f_{NL} = f_{NL}(k_1, k_2, k_3)$
- 4-point correlator $\Rightarrow$ trispectrum with g_{NL} (connected part) and $\tau_{NL} \sim f_{NL}^2$ (disconnected part)

Curvaton models

- Standard scenario: inflaton fluctuates $\Rightarrow$ adiabatic perturbation $\Rightarrow$ imprinted on decay products
- All light fields fluctuate during inflation $\Rightarrow$ isocurvature perturbation
- If some field is long-lived, its energy density may become the dominant energy form $\Rightarrow$ at the decay, the isocurvature perturbation is converted into adiabatic perturbation
- if inflaton perturbations are small enough, the curvature perturbation will be due to curvaton alone

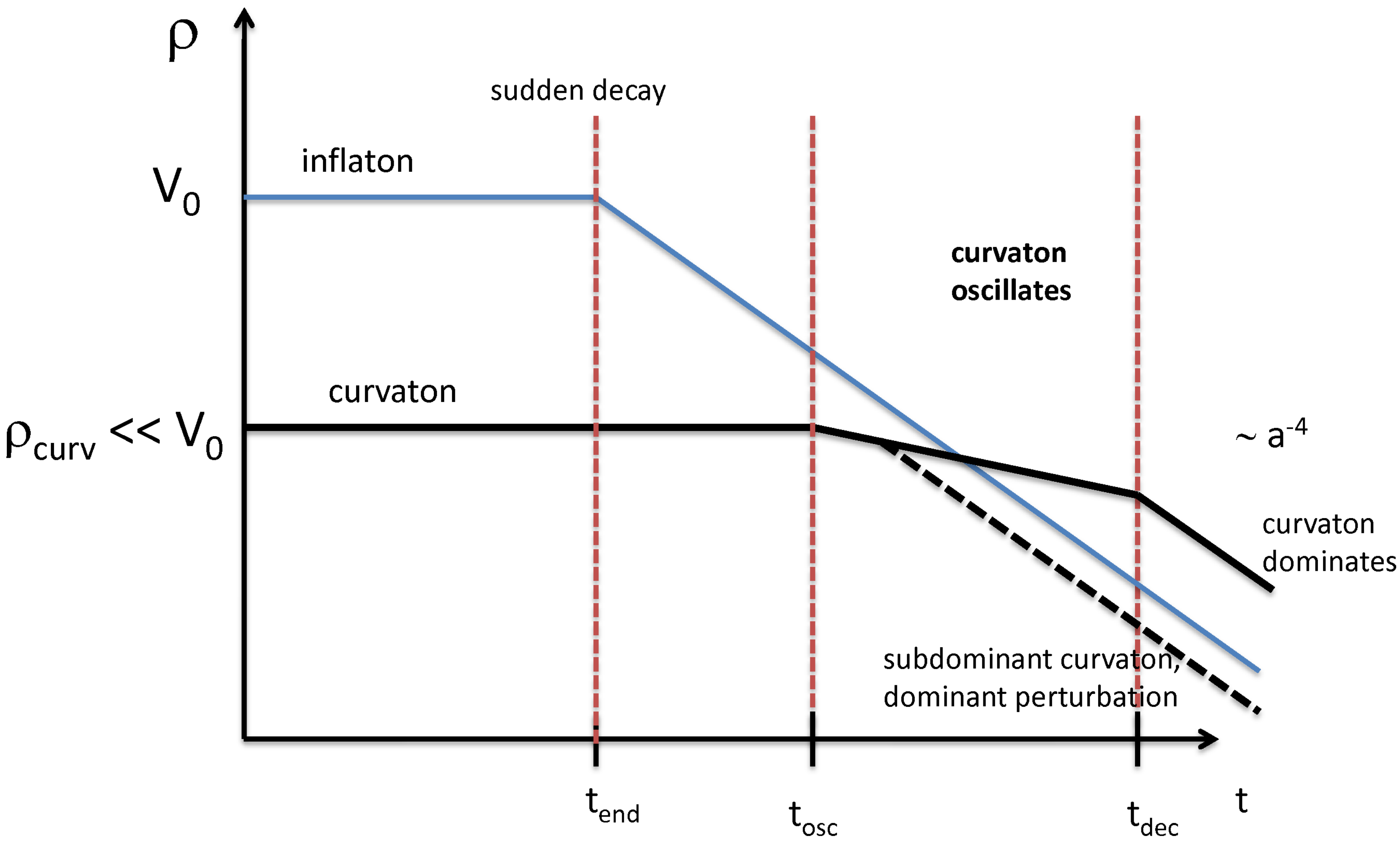


Figure 5: Curvaton model schematically

The effectively massless curvaton field σ has the spectrum

$$P_\sigma \simeq \langle(\delta\sigma)^2\rangle = \left(\frac{H_*}{2\pi}\right)^2, \quad m_\sigma \ll H_* \tag{117}$$

where H_* is the Hubble rate during inflation and $\rho_\sigma \ll \rho_{inf}$. The initial value is σ_*. After inflation curvaton starts to oscillate when

$$H \sim H_*/t \sim m_\sigma \tag{118}$$

For quadratic potential σ and $\delta\sigma$ evolve in the same way:

$$\ddot{\sigma} + m_\sigma^2\sigma = 0 \Rightarrow \delta\ddot{\sigma} + m_\sigma^2\delta\sigma = 0 \tag{119}$$

Thus

$$\frac{\delta\sigma}{\sigma^2} = \frac{\delta\sigma_*}{\sigma_*^2} \simeq \left(\frac{H_*}{\sigma_*}\right)^2 \tag{120}$$

Curvaton energy density is

$$\rho_\sigma = \frac{1}{2}\dot{\sigma}^2 + \frac{1}{2}m_\sigma^2 = \frac{1}{2}m_\sigma^2\sigma_0^2 \ , \tag{121}$$

where σ_0 is the amplitude of the oscillations. Thus the energy density perturbation

$$\delta \equiv \frac{\delta\rho_\sigma}{\rho_\sigma} = 2\frac{\delta\sigma}{\sigma} \tag{122}$$

and the curvature perturbation due to the curvaton is

$$\zeta = -H\frac{\delta\rho_\sigma}{\dot{\rho}_\sigma} = \frac{1}{3}\frac{\delta\rho_\sigma}{\rho_\sigma} \equiv \frac{1}{3}\delta \tag{123}$$

since $\rho_\sigma \sim a^{-3}$.

The total curvature perturbation is

$$\zeta = -H\frac{\delta\rho}{\dot{\rho}} = \frac{3\rho_\sigma\zeta_\sigma}{4\rho_r + 3\rho_\sigma} \tag{124}$$

since for radiation $\rho_r \sim a^{-4}$ and we assume that $\delta\rho_r \simeq 0$. Thus we may write

$$\zeta = \frac{\rho_\sigma}{4\rho_r + 3\rho_\sigma}\delta \equiv \frac{1}{4}r\delta \tag{125}$$

where r is now the fraction

$$r = \frac{4\rho_\sigma}{4\rho_r + 3\rho_\sigma} \simeq \frac{\rho_\sigma}{\rho_r} \text{ if } r \ll 1. \tag{126}$$

Non-gaussianity in curvaton models

The number of e-folds between the time the curvaton starts to oscillate and its decay is

$$N = \ln \frac{a_{osc}}{a_{dec}} = \frac{1}{3} \ln \frac{\frac{1}{2} m_\sigma^2 \sigma_*^2}{\rho_{dec}(\sigma)} \tag{127}$$

and

$$\begin{aligned} N' &= \frac{\partial N}{\partial \sigma_*} = \frac{r}{2\sigma_*} \\ N'' &= \frac{1}{\sigma_*^2} \left(-\frac{8}{3} r^3 - r^2 + r \right) \end{aligned} \tag{128}$$

The non-gaussianity parameter in the curvaton model is thus

$$-\frac{3}{5} f_{NL} = \frac{1}{2} \frac{N''}{(N')^2} \Rightarrow f_{NL} = \frac{5}{3} + \frac{5}{8} r - \frac{5}{3r} \tag{129}$$

- large non-gaussianity when $r \to 0 \Rightarrow$ constraint on r.

Computing N': since for radiation $\rho \sim a^{-4}$, write

$$N = \frac{1}{4} \ln \frac{\rho_r(t_*)}{\rho_r(t_{dec})} \tag{130}$$

while

$$\rho_\sigma(t_{dec}) = \frac{1}{2} m_\sigma^2 \sigma_*^2 \left(\frac{\rho_r(t_*)}{\rho(t_{dec})} \right)^{3/4} \tag{131}$$

where $\rho(t_{dec}) = \rho_r(t_{dec}) + \rho_\sigma(t_{dec})$ and the decay takes place on a constant density hypersurface, i.e. total density does not depend on σ_*:

$$\partial_\sigma \rho(t_{dec}) = 0 \Rightarrow \rho'_\sigma = -\rho'_r \; . \tag{132}$$

Likewise, since the curvaton energy is negligible at the end of inflation,

$$\partial_\sigma \rho_r(t_*) = 0 \; . \tag{133}$$

Thus

$$N' = -\frac{1}{4} \frac{\rho'_r(t_{dec})}{\rho_r(t_{dec})} = \frac{1}{4} \frac{\rho'_\sigma(t_{dec})}{\rho_r(t_{dec})} \tag{134}$$

We also find that

$$\rho'_\sigma = \frac{\rho'_\sigma}{2\sigma_*} + \frac{3}{4} \rho_\sigma \frac{\rho'_r}{\rho_r} \Rightarrow \rho'_\sigma = \frac{4\rho_\sigma \rho_r}{4\rho_r + 3\rho_\sigma} \frac{1}{2\sigma_*} \tag{135}$$

so that

$$N' = \frac{r}{2\sigma_*} \tag{136}$$

- curvaton interactions can change the results
- field oscillations in the non-quadratic regime→nonlinearites
- e.g. $f_{NL} \simeq 0$ possible

$$V = \frac{1}{2}m^2\sigma^2 + \frac{\sigma^8}{M^4}$$