Path Integral Monte Carlo Simulations of Warm Dense Matter



Plan for L5, L9, L12 and T1 & T3

- L5: Path integral Monte Carlo (PIMC) simulations and First Principles Equation of state (FPEOS) database
- L9: NASA mission Juno to Jupiter, dilute core
- **T1: "Build that Planet" with SPH method**

L12: NASA mission Cassini to Saturn. How did that planet become the Lord of the Rings?

T3: FPEOS tutorial

Software needed for T1 & T3

T1: "Built that Planet" with SPH method Python + Jupyter notebooks (installation on laptop required, for example with Anaconda) **Alternative: use Google Colab (no installation,** no animation) T3: FPEOS tutorial **Requires a C++ compiler for all calculations,**

uses Python for all graphics.

Outline of lecture 1

- Path integral Monte Carlo (PIMC) method
- Comparison with different experiments
- First Principles Equation of state (FPEOS) database



Center for Matter under Extreme Conditions (CMEC)







Study planetary interiors in the laboratory: shock wave experiments



Two-stage gas gun (Livermore) 0.2 Mbar



Nova laser (Livermore) 3.4 Mbar



Z capacitor bank (Sandia) 2 Mbar



National Ignition Facility 700 Mbar

Shock wave measurements determine the Equation of State on the Hugoniot curve



Shock wave measurements determine the Equation of State on the Hugoniot curve











Path Integral Monte Carlo

Density functional molecular dynamics at lower T





Born-Oppenheimer approx. MD with classical nuclei:

F = m a

Forces derived DFT with electrons in the instantaneous ground state.

Path integral Monte Carlo at high T > 10⁴...10⁶ K







Starting from Restricted PIMC Simulations of Hydrogen

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Equation of State of the Hydrogen Plasma by Path Integral Monte Carlo Simulation

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Molecular Disconiation in Hot. Dance Hudrogen									
Wolecular Dissociation in 1100, Dense Hydrogen									
W.R. Magro, ¹ D.M. Ceperley, ² C. Pierleoni, ³ and B. Bernu ⁴									

Canonical Ensembles: Classical

Boltzmann factor

$$e^{-E / k_B T}$$

Thermodynamic averages:

$$Z_{Cl} = \sum_{S} e^{-\beta E_{S}}$$

Canonical Ensembles: Classical Quantum **Boltzmann factor Density matrix** $e^{-E / k_B T}$ $\hat{\rho} = e^{-\beta \hat{H}}$ $\rho(R,R',\beta) = \langle R | e^{-\beta \hat{H}} | R' \rangle$ $\rho(R,R',\beta) = \sum_{S} e^{-\beta E_{S}} \Psi_{S}^{*}(R) \Psi_{S}(R')$

Thermodynamic averages:

$$Z_{Cl} = \sum_{S} e^{-\beta E_{S}}$$

$$Z_{Q} = Tr[\hat{\rho}] = \int dR \, \langle R | e^{-\beta \hat{H}} | R \rangle$$
$$\langle \hat{O} \rangle = \frac{Tr[\hat{O} \hat{\rho}]}{Tr[\hat{\rho}]}$$

Step 1 towards the path integral

Matrix squaring property of the density matrix

Matrix squaring in operator notation:

$$\hat{\rho} = e^{-\beta \hat{H}} = \left(e^{-(\beta/2)\hat{H}}\right) \left(e^{-(\beta/2)\hat{H}}\right), \quad \beta = \frac{1}{k_B T}$$

Matrix squaring in real-space notation:

$$\langle R \mid \hat{\rho} \mid R' \rangle = \int dR_1 \, \langle R \mid e^{-(\beta/2)\hat{H}} \mid R_1 \rangle \, \langle R_1 \mid e^{-(\beta/2)\hat{H}} \mid R' \rangle$$

Matrix squaring in matrix notation:

$$\begin{bmatrix} \dots & R' & \dots \\ R & \ddots & \vdots \\ \dots & \dots & \dots \end{bmatrix} = \begin{bmatrix} \dots & R_1 & \dots \\ R & \ddots & \vdots \\ \dots & \dots & \dots \end{bmatrix} * \begin{bmatrix} \dots & R' & \dots \\ R_1 & \ddots & \vdots \\ \dots & \dots & \dots \end{bmatrix}$$

Repeat the matrix squaring step

Matrix squaring in operator notation:

$$\hat{\rho} = e^{-\beta \hat{H}} = \left(e^{-(\beta/4)\hat{H}}\right)^4, \quad \beta = \frac{1}{k_B T}$$

Matrix squaring in real-space notation:

$$\langle R \mid \hat{\rho} \mid R' \rangle = \int dR_1 \int dR_2 \int dR_3 \langle R \mid e^{-(\beta/4)\hat{H}} \mid R_1 \rangle \langle R_1 \mid e^{-(\beta/4)\hat{H}} \mid R_2 \rangle \langle R_2 \mid e^{-(\beta/4)\hat{H}} \mid R_3 \rangle \langle R_3 \mid e^{-(\beta/4)\hat{H}} \mid R' \rangle$$

Path Integrals in Imaginary Time

Every particle is represented by a path, a ring polymer.

Density matrix:

$$\hat{\rho} = e^{-\beta \hat{H}} = \left(e^{-\tau \hat{H}}\right)^M, \ \beta = \frac{1}{k_B T}, \ \tau = \frac{\beta}{M}$$

$$\langle \hat{O} \rangle = \frac{\text{Tr}[\hat{O}\hat{\rho}]}{\text{Tr}[\hat{\rho}]}$$

Trotter break-up:

$$\left\langle R \mid \hat{\rho} \mid R' \right\rangle = \left\langle R \mid (e^{-\tau \hat{H}})^{M} \mid R' \right\rangle = \int dR_{1} \dots \int dR_{M-1} \left\langle R \mid e^{-\tau \hat{H}} \mid R_{1} \right\rangle \left\langle R_{1} \mid e^{-\tau \hat{H}} \mid R_{2} \right\rangle \dots \left\langle R_{M-1} \mid e^{-\tau \hat{H}} \mid R' \right\rangle$$

Path Integrals in Imaginary Time Simplest form for the paths' action: primitive approx.

Density matrix:

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۶ V(r₁₂)

Trotter break-up:

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Trotter formula:

$$e^{-\beta(\hat{T}+\hat{V})} = \lim_{M \to \infty} \left[e^{-\tau \, \hat{T}} e^{-\tau \, \hat{V}} \right]^{M}$$

Path integral and primitive action S:

$$\langle R \mid \hat{\rho} \mid R' \rangle = \oint_{R \to R'} dR_t e^{-S[R_t]}$$

$$S[R_t] = \sum_{i=1}^M \frac{(R_{i+1} - R_i)^2}{4\lambda\tau} + \frac{\tau}{2} \Big[V(R_i) + V(R_{i+1}) \Big]$$

Pair action: Militzer, Comp. Phys. Comm. (2016)





Douglas Adams: *"Infinite Improbability Drive"* of spaceship *"Heart of Gold"*



Path Integrals in Imaginary Time include Zero-Point Motion and some Tunnelling Effects

$$\langle R \mid \hat{\rho} \mid R' \rangle = \oint_{R \to R'} dR_t e^{-S[R_t]} \\ S[R_t] = \sum_{i=1}^{M} \frac{(R_{i+1} - R_i)^2}{4\lambda\tau} + \frac{\tau}{2} \Big[V(R_i) + V(R_{i+1}) \Big]$$





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Bosonic and Fermionic Density Matrices

Bosonic density matrix: Sum over all symmetric eigenstates.

$$\rho_{B}(R,R',\beta) = \sum_{i} e^{-\beta E_{i}} \Psi_{S}^{[i]^{*}}(R) \Psi_{S}^{[i]}(R')$$

Fermionic density matrix: Sum over all antisymmetric eigenstates.

$$\rho_F(R, R', \beta) = \sum_i e^{-\beta E_i} \Psi_{AS}^{[i]*}(R) \Psi_{AS}^{[i]}(R')$$

Bosonic and Fermionic Path Integrals

Bosonic density matrix: Sum over all symmetric eigenstates.

$$\rho_{B}(R,R',\beta) = \sum_{i} e^{-\beta E_{i}} \Psi_{S}^{[i]*}(R) \Psi_{S}^{[i]}(R')$$

Project out the symmetric states:

$$\rho_{B}(R,R',\beta) = \sum_{P} (+1)^{P} \rho_{D}(R,PR',\beta)$$

Fermionic density matrix: Sum over all antisymmetric eigenstates.

$$\rho_F(R,R',\beta) = \sum_i e^{-\beta E_i} \Psi_{AS}^{[i]*}(R) \Psi_{AS}^{[i]}(R')$$

Project out the antisymmetric states:

$$\rho_F(R,R',\beta) = \sum_P (-1)^P \rho_D(R,PR',\beta)$$

$$\left\langle R \mid \hat{\rho}_{F/B} \mid R' \right\rangle = \sum_{P} (\pm 1)^{P} \int dR_{1} \dots \int dR_{M-1} \left\langle R \mid e^{-\tau \hat{H}} \mid R_{1} \right\rangle \dots \left\langle R_{M-1} \mid e^{-\tau \hat{H}} \mid PR' \right\rangle$$

Bosonic and Fermionic Path Integrals

Bosonic density matrix: Sum over all symmetric eigenstates.

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Restricted PIMC for fermions: How is the restriction applied?



Free-particle nodes:

Construct a <u>fermionic trial density matrix</u> in form of a Slater determinant of single-particle density matrices:

$$\rho_T(R,R',\beta) = \begin{vmatrix} \rho(r_1,r_1',\beta) & \cdots & \rho(r_1,r_N',\beta) \\ \vdots & \ddots & \vdots \\ \rho(r_N,r_1',\beta) & \cdots & \rho(r_N,r_N',\beta) \end{vmatrix}$$

Enforce the following nodal condition for all time slices along the paths:

$$\rho_T[R(t), R(0), t] > 0$$

This 3N-dimensional conditions eliminates all negative and some positive contribution to the path \rightarrow Solves the fermion sign problem approx.

$$\rho_0^{[1]}(r,r';\beta) = \sum_k e^{-\beta E_k} \, \Psi_k(r) \, \Psi_k^*(r')$$

Fermionic Path Integrals Example: Closed paths of 2 free particles

Distinguishable particles:

Consider path types: NA + NX

Bosons:

Consider path types: NA + NX + PNX

Direct fermions:

Consider path types: NA + NX - PNX

Restricted fermions:

Consider only path type: NA





Molecular Hydrogen

Snapshot from a PIMC simulation with 32 protons and electrons







100% molecules, weakly interacting

Molecular Hydrogen

Snapshot from a PIMC simulation with 32 protons and electrons



2 protons (pink spheres) and spin-up and one spin-down electron form one H_2 molecule.



- strongly interacting molecules, close to pressure dissociation
- Electrons are degenerate, partially delocalized
- Electron paths are permuting

Metallic Hydrogen

Snapshot from a PIMC simulation with 32 protons and electrons



Free protons (pink spheres) and delocalized electrons.



- Pressure dissociation, free protons
- Degenerate electron gas
- High number of permutations

Silicates: MgSiO₃

MgSiO₃ : Principal Hugoniot Curve



Gonzalez, Soubiran, Peterson, Militzer, *Phys. Rev. B* **101** (2020) 024107

MgSiO₃ : Principal Hugoniot Curve



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Gonzalez, Soubiran, Peterson, Militzer, Phys. Rev. B 101 (2020) 024107



Inertial confinement fusion experiments with plastic coated spheres of liquid H₂



(Graphics: Bachmann et al. LLNL)

PIMC and DFT-MD simulations performed for C₂H, CH, C₂H₃, CH₃ and CH₄.











Hugoniot Curves of BN and B₄C Fully interacting EOS and Linear Mixing agree quite well.



Hugoniot Curves of BN and B₄C Fully interacting EOS and Linear Mixing agree quite well.



Linear Mixing at Constant P and T (Also called additive volume rule)





$$V_{
m mix} = N_1 V_1 + N_2 V_2 ,$$

 $m_{
m mix} = N_1 m_1 + N_2 m_2 ,$
 $E_{
m mix} = N_1 E_1 + N_2 E_2 ,$
 $\rho_{
m mix} = m_{
m mix} / V_{
m mix}$

Hugoniot Curves of MgO and MgSiO₃ Results from fully interacting EOS and experiment.



Hugoniot Curves of MgO and MgSiO₃ Fully interacting EOS and Linear Mixing agree quite well.



Linear mixing works well for $T \gtrsim 2 \times 10^5$ K and $\varrho/\varrho_0 \gtrsim 3.2$

The Journal of Chemical Physics

Nonideal mixing effects in warm dense matter studied with first-principles computer simulations



Nonlinear Mixing Effects in MgSiO₃ Fully interacting EOS and Linear Mixing agree quite well.



Linear mixing works well for $T \gtrsim 2 \times 10^5$ K and $\varrho/\varrho_0 \gtrsim 3.2$

Hugoniot Curves of CO and CO₂

Experimental CO₂ Hugoniot agree with Linear Mixing result



Hugoniot Curves of H₂O, H₂O₂, and Al₂O₃ Experimental H₂O Hugoniot agree with Linear Mixing result





Material	Number	Minimum	Maximum	Minimum	Maximum	Number of	References
	of isochores	density	density	temperature	temperature	EOS points	
		$[\mathrm{gcm^{-3}}]$	$[\mathrm{gcm^{-3}}]$	[K]	[K]		
Hydrogen	33	0.001	798.913	15625	6.400×10^{7}	401	[69–74]
Helium	9	0.387	10.457	500	$2.048{ imes}10^9$	228	[75, 76]
Boron	16	0.247	49.303	2000	$5.174 { imes} 10^{8}$	314	[77]
Carbon	9	0.100	25.832	5000	$1.035{ imes}10^9$	162	[78, 79]
Nitrogen	17	1.500	13.946	1000	$1.035{ imes}10^9$	234	[80]
Oxygen	6	2.486	100.019	10000	$1.035{ imes}10^9$	76	[81]
Neon	4	0.895	15.026	1000	$1.035{ imes}10^9$	67	[82]
Sodium	9	1.933	11.600	1000	$1.293{ imes}10^{8}$	193	[83, 84]
Magnesium	23	0.431	86.110	20000	5.174×10^{8}	371	[85]
Aluminum	15	0.270	32.383	10000	$2.156{ imes}10^8$	240	[86]
Silicon	7	2.329	18.632	50000	$1.293{ imes}10^8$	85	[87, 88]
LiF	8	2.082	15.701	10000	1.035×10^{9}	91	[89]
B_4C	16	0.251	50.174	2000	$5.174 { imes} 10^{8}$	291	[90]
BN	16	0.226	45.161	2000	5.174×10^{8}	311	[91]
CH_4	16	0.072	14.376	6736	$1.293{ imes}10^8$	247	[92, 93]
CH_2	16	0.088	17.598	6736	$1.293{ imes}10^{8}$	248	[92, 93]
C_2H_3	16	0.097	19.389	6736	$1.293{ imes}10^{8}$	247	[92, 93]
CH	16	0.105	21.000	6736	$1.293{ imes}10^{8}$	248	[92, 93]
C_2H	16	0.112	22.430	6736	$1.293{ imes}10^{8}$	245	[92, 93]
MgO	19	0.357	71.397	20000	5.174×10^{8}	286	[94]
$MgSiO_3$	16	0.321	64.158	6736	5.174×10^{8}	284	[95, 96]

First-Principles Equation of State Database online http://militzer.berkeley.edu/FPEOS



5000 first-principles calculations have been combined into our FPEOS database. So anyone can predict shock Hugoniot curves for a variety of compounds and mixtures. This will make warm dense matter calculations more reliable and efficient.

NIF Gbar Experiment: Equations of State of C-O Mixtures in White Dwarf Stars

PI: D. Saumon (LANL), Blouin, Glenzer, Swift, Kritcher, Doppner, Whitley, Lazicki, Falcone, Militzer

We propose to make EOS measurements along the Hugoniot with the Gbar platform of carbon-oxygen rich materials that resemble conditions in White Dwarf stars.



Glyoxal C₂0₂H₂ comp:~/fpeos> fpeos binaryMixture EOS1=6 2.0 EOS2=18 2.0 rho0=1.27 E0=-227.8 Acetic acid C₂0₂H₄ comp:~/fpeos> fpeos binaryMixture EOS1=6 2.0 EOS2=16 2.0 rho0=1.049 E0=-229.0

FPEOS demo

The End