Multi-body final states production in electron-positron annihilation and their contributions to $(g\text{-}2)_\mu$

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Based on: arxiv: 2403.14294, JHEP07 (2023) 037,
RPP84 (2021) 076201, JHEP03 (2021) 092,
PRD99 (2019) 114015, PRD97 (2018) 036012,
PRD95 (2017) 056007, PRD94 (2016) 116061,
PRD90 (2014) 036004, PLB736 (2014) 11,
PRD88 (2013) 056001, et. al.

MIP 2024

Apr 2024, Beijing

湖南大学

HUNAN UNIVERSITY

Outlines

1. Introduction
2. Framework：RChT
3. HVP
4. Summary

Introduction: muon g-2

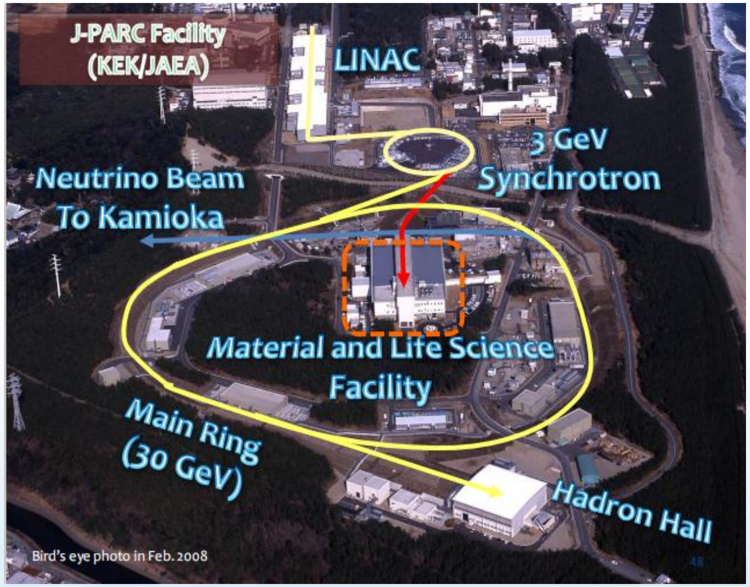
- Why muon? life time is long: 2.2 μs, τ---$2.9 \times 10^{-7} \mu s$
- Sensitive to new physics (M^2)

439 rounds in Fermi's ring !

- Elementary particle: g is close to 2.
  - Electron: g=2.00231930436152(56) [PDG2022], close to theoretical prediction $g = 2[1 + \frac{\alpha}{4\pi} + O(\alpha^2)]$
  - Composite particle: g=5.6 for proton and g=-3.8 for neutron.

See also Liang Li's talk at Hunan university

- Muon g-2: one of the most precise indicator of new physics



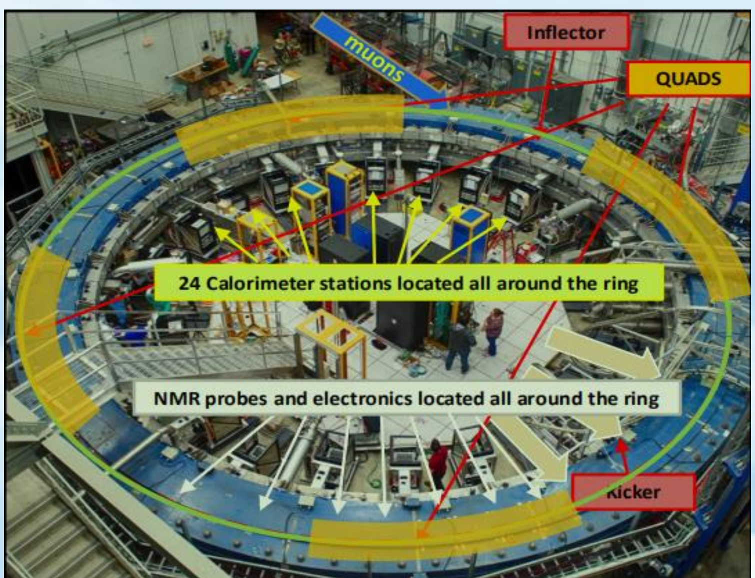
J-PARC

BNL E821 J-PARC E3

g-2: 0.46 ppm → 0.37 ppm (→0.1ppm)

50 times of number of events as large as BNL's to 0.46ppm

2001，2009，2025？



FNAL

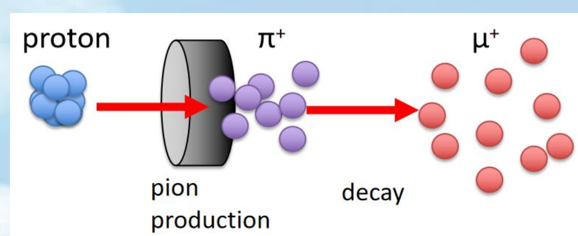
Run1: only 6% of full statistics used now

Run2-3: analyzing, factor 2 improvment

Run4: 13 times as large as BNL's

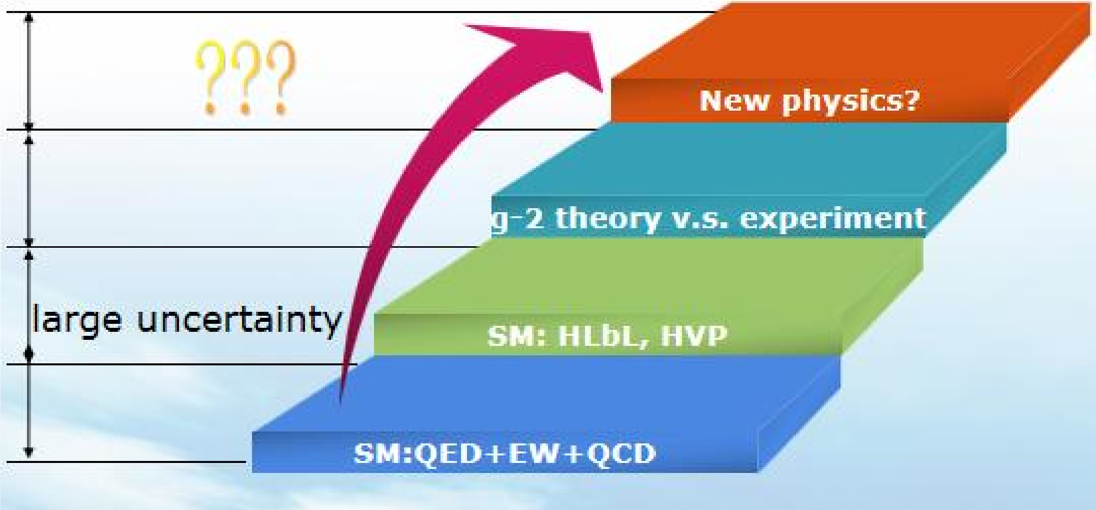
Run5: 20 times as large as BNL's

2017, 2021, 2023……



CSNS,
HIAF?

uncertainty from SM



$$a_\mu = a_\mu^{\rm QED} + a_\mu^{\rm EW} + a_\mu^{\rm QCD}$$

- HVP, HLbL?

	values ($\times 10^{-11}$)
QED	116584718.931(104)
EW	153.6(1.0)
HVP	6845(40)
HLBL	92(18)
SM	116591810(43)
exp.(BNL)	116592089(63)
exp.(FNAL)	116592040(54)
exp.(avg.)	116592061(41)
$a_\mu^{\rm SM}$-$a_\mu^{\rm exp}$	251(59)

Phys.Rev.Lett.126, 141801 (2021)
Phys.Rev.D 73, 072003 (2006).

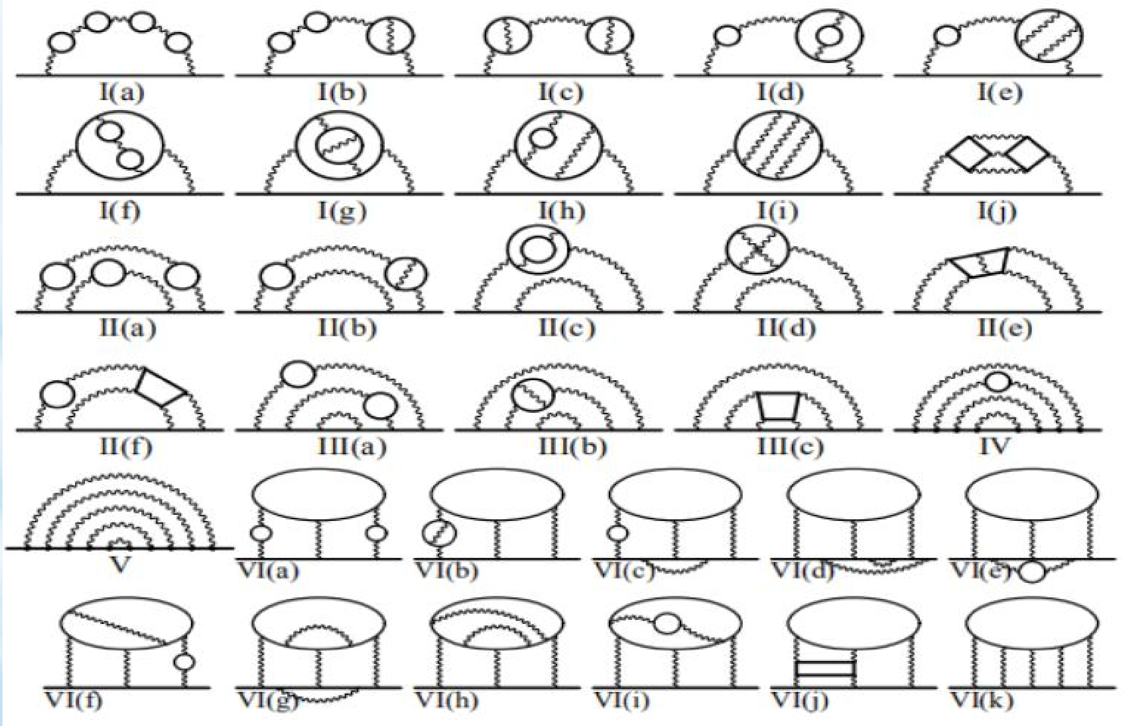
Phys.Rept.887(2020)1

Science and Technology Cooperation Program in High Energy Physics. This review benefited from discussions with O. Catà, N. Christ, L.Y. Dai, H. Davoudiasl, S. Fayer, S. Ganguly, A. Gasparian, S. Hashimoto, T. Iijima, K. Kampf, D. Kawall, I. Larin, Z. Pagel, M. Petschlies, A. Rebhan, K. Schilcher, K. Shimomura, E. Shintani, D. Steffen, S. Tracz, C. Tu, and T. Yamazaki.

QED

- The most contribution
- Precise prediction
- At 10-th order, $O(\alpha^5)$

$$a_\mu = 116\,584\,718.951\,(0.080) \times 10^{-11}$$



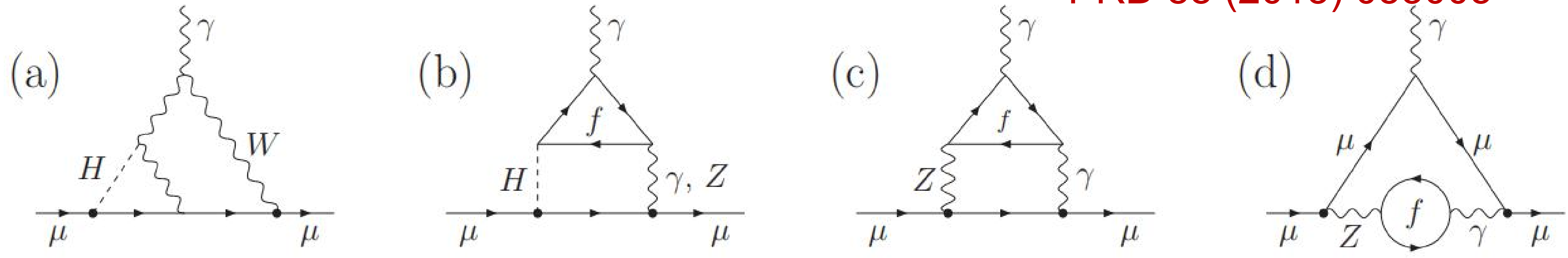
Aoyama *et.al.*,
PRL109 (2012) 111808

EW+Strong interactions

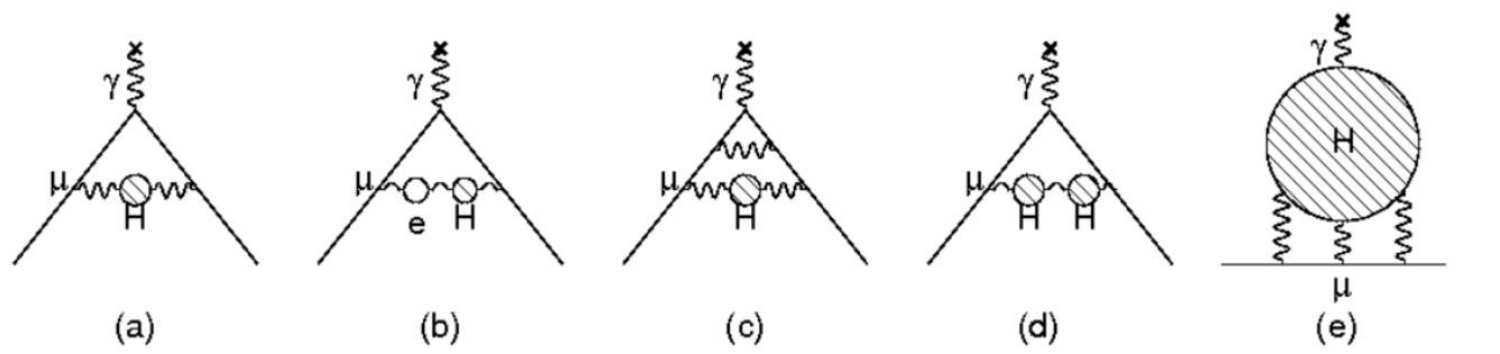
- Precise prediction
- At two-loop level

$$a_\mu = 153.6\,(1.0)\ \times\ 10^{-11}$$

Gnendiger *et.al.*,
PRD 88 (2013) 053005



- Strong interactions: pQCD---high energy region



Phys.Rept.887(2020)1

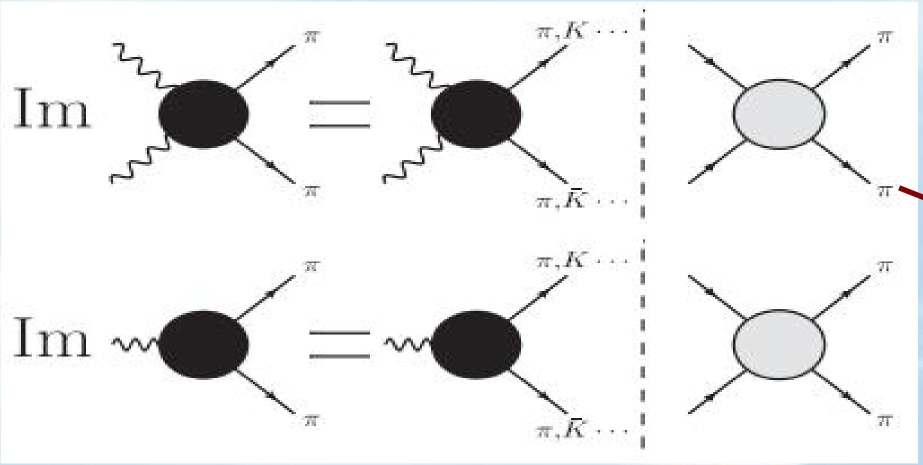
2. Framework

- Hadronic Part: Methods from SM
  - LQCD
  - Data-driven solutions from experiment
  - Amplitude analysis: model independent

- Only one physical amplitude!
- It should satisfy the fundamental QFT principles
- It should be compatible with the exp results

Amplitude analysis: FSI

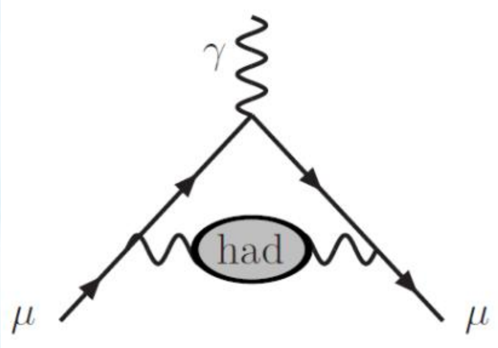
- Most resonances decays into light pseudoscalars
- FSI needs to be taken into account to perform an amplitude analysis
- Methods: KM, N/D, AMP, Roy equation, PKU, Pade, LSE, BSE, ChEFT, *et.al.*



Yao, Dai#, Zheng, Zhou, RPP84(2021)076201

Different energy regions

- QCD: high energy region
- Dispersive approach: Roy, KT, PKU, etc., difficult to deal with multi-body rescattering
- ChPT: works in the very low energy region
- RChT: extend to a bit higher energy region



$$a_\mu^{\mathrm{had}} = \left(\frac{\alpha_e(0) m_\mu}{3\pi}\right)^2 \int_{s_{\mathrm{thr}}}^{\infty} \mathrm{d}s \frac{\hat{K}(s)}{s^2} R_{\mathrm{h}}(s)$$

Low energy physics dominates

RChT

- Resonances included as new degrees of freedom

$$R \equiv \frac{1}{\sqrt{2}} \sum_{i=1}^{8} \lambda_i \phi_R^i$$

- Construct Lagrangians by discrete and chiral symmetries

$$\mathcal{L}_{\rm kin}^R = -\frac{1}{2}\langle \nabla^\lambda R_{\lambda\mu} \nabla_\nu R^{\nu\mu} \rangle + \frac{M_R^2}{4}\langle R_{\mu\nu} R^{\mu\nu} \rangle , \qquad R = V, A ,$$

$$\mathcal{L}_{\rm kin}^R = \frac{1}{2}\langle \nabla^\mu R \nabla_\nu R - M_R^2 R^2 \rangle , \qquad R = S, P .$$

$$\mathcal{L}_{(4)}^R = \sum_{i=1}^{22} \lambda_i^V \mathcal{O}_i^V + \sum_{i=1}^{17} \lambda_i^A \mathcal{O}_i^A + \sum_{i=1}^{18} \lambda_i^S \mathcal{O}_i^S + \sum_{i=1}^{13} \lambda_i^P \mathcal{O}_i^P$$

$$\mathcal{L}_{(2)}^{RR} = \sum_{(ij)n} \lambda_n^{R_i R_j} \mathcal{O}_n^{R_i R_j} ,$$

$$\mathcal{L}_{(0)}^{RRR} = \sum_{(ijk)} \lambda^{R_i R_j R_k} \mathcal{O}^{R_i R_j R_k} .$$

i	Operator $\mathcal{O}_i^{RR}$, $R = V, A$	Operator $\mathcal{O}_i^{SS}$	Operator $\mathcal{O}_i^{PP}$
1	$\langle R_{\mu\nu} R^{\mu\nu} u^\alpha u_\alpha \rangle$	$\langle S S u_\mu u^\mu \rangle$	$\langle P P u_\mu u^\mu \rangle$
2	$\langle R_{\mu\nu} u^\alpha R^{\mu\nu} u_\alpha \rangle$	$\langle S u_\mu S u^\mu \rangle$	$\langle P u_\mu P u^\mu \rangle$
3	$\langle R_{\mu\alpha} R^{\nu\alpha} u^\mu u_\nu \rangle$	$\langle S S \chi_+ \rangle$	$\langle P P \chi_+ \rangle$
4	$\langle R_{\mu\alpha} R^{\nu\alpha} u_\nu u^\mu \rangle$		
5	$\langle R_{\mu\alpha} (u^\alpha R^{\mu\beta} u_\beta + u_\beta R^{\mu\beta} u^\alpha) \rangle$		
6	$\langle R_{\mu\nu} R^{\mu\nu} \chi_+ \rangle$		
7	$i \langle R_{\mu\alpha} R^{\alpha\nu} f_{+\beta\nu} \rangle g^{\beta\mu}$		

Tensors

- Tensors included as new degrees of freedom

$$T_{\mu\nu} = \begin{pmatrix} \frac{a_2^0}{\sqrt{2}} + \frac{f_2^8}{\sqrt{6}} + \frac{f_2^0}{\sqrt{3}} & a_2^+ & K_2^{*+} \\ a_2^- & -\frac{a_2^0}{\sqrt{2}} + \frac{f_2^8}{\sqrt{6}} + \frac{f_2^0}{\sqrt{3}} & K_2^{*0} \\ K_2^{*-} & \bar{K}_2^{*0} & -\frac{2f_2^8}{\sqrt{6}} + \frac{f_2^0}{\sqrt{3}} \end{pmatrix}_{\mu\nu}$$

- Effective Lagrangians

$$\begin{aligned} \mathcal{O}^1_{\rm TJP} &= i\varepsilon_{\mu\nu\rho\sigma}\langle[T^{\mu\alpha}, f_+^{\rho\sigma}]\nabla_\alpha u^\nu\rangle, \\ \mathcal{O}^2_{\rm TJP} &= i\varepsilon_{\mu\nu\rho\sigma}\langle[\nabla^\nu T^\mu_\alpha, f_+^{\rho\sigma}]u^\alpha\rangle, \\ \mathcal{O}^3_{\rm TJP} &= i\varepsilon_{\mu\nu\rho\sigma}\langle[\nabla^\nu T^\mu_\alpha, f_+^{\rho\alpha}]u^\sigma\rangle. \end{aligned}$$

$$\begin{aligned} \mathcal{O}^1_{\rm TVP} &= i\varepsilon_{\mu\nu\rho\sigma}\langle[T^{\mu\alpha}, V^{\rho\sigma}]\nabla_\alpha u^\nu\rangle, \\ \mathcal{O}^2_{\rm TVP} &= i\varepsilon_{\mu\nu\rho\sigma}\langle[\nabla^\nu T^\mu_\alpha, V^{\rho\sigma}]u^\alpha\rangle, \\ \mathcal{O}^3_{\rm TVP} &= i\varepsilon_{\mu\nu\rho\sigma}\langle[\nabla^\nu T^\mu_\alpha, V^{\rho\alpha}]u^\sigma\rangle. \end{aligned}$$

- Linearly independent terms
  - Equations of motion
  - Total derivative
  - Schouten identity

$$\nabla_\mu u^\mu = \frac{i}{2}\left(\chi_- - \frac{1}{n_f}\langle\chi_-\rangle\right)$$

$$\langle\nabla_\mu (ABC\cdots)\rangle = \langle(\nabla_\mu A)BC\cdots\rangle + \langle A(\nabla_\mu B)C\cdots\rangle + \langle AB(\nabla_\mu C)\cdots\rangle + \cdots$$

$$g_{\alpha\lambda}\varepsilon_{\mu\nu\rho\sigma} + g_{\alpha\mu}\varepsilon_{\nu\rho\sigma\lambda} + g_{\alpha\nu}\varepsilon_{\rho\sigma\lambda\mu} + g_{\alpha\rho}\varepsilon_{\sigma\lambda\mu\nu} + g_{\alpha\sigma}\varepsilon_{\lambda\mu\nu\rho} = 0$$

Power-counting

- 1/Nc expansion,
  - Loop diagrams are suppressed
  - Uncertainty ~1/3

- 'Chiral counting' by integrating out resonances
  - Those generating O(p^6) ChPT Lagrangians

$$\langle R_a \chi(p^4) \rangle, \langle R_a R_b \chi(p^2) \rangle \text{ and } \langle R_a R_b R_c \rangle.$$

Dai *et.al.*, PRD99 (2019) 114015

Matching GF: reduce LECs

- Matching GF between QCD and ChEFT in the high energy region, using large Nc and OPE.

$$\left(\Pi^{ijk}_{SAA}\right)_{\mu\nu} = i^2 \int d^4x\, d^4y\, e^{i(p_1\cdot x+p_2\cdot y)} \langle 0|T\left\{S^i(0)A^j_\mu(x)A^k_\nu(y)\right\}|0\rangle$$

$$\left(\Pi^{ijk}_{SVV}\right)_{\mu\nu} = i^2 \int d^4x\, d^4y\, e^{i(p_1\cdot x+p_2\cdot y)} \langle 0|T\left\{S^i(0)V^j_\mu(x)V^k_\nu(y)\right\}|0\rangle$$

$$S^i(x) = \left(\bar{q}\lambda^i q\right)(x) \qquad V^i_\mu(x) = \left(\bar{q}\gamma_\mu\frac{\lambda^i}{2}q\right)(x) \qquad A^i_\mu(x) = \left(\bar{q}\gamma_\mu\gamma_5\frac{\lambda^i}{2}q\right)(x)$$

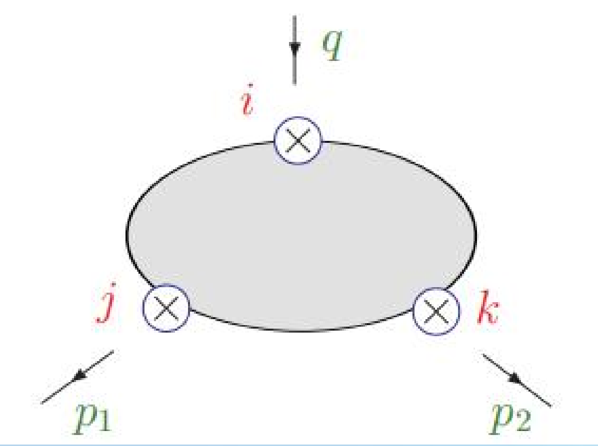
- Ward identity

$$p_1^\mu\left(\Pi^{ijk}_{SAA}\right)_{\mu\nu} = -2\,d^{ijk}B_0F^2\frac{(p_2)_\nu}{p_2^2}$$

$$p_2^\nu\left(\Pi^{ijk}_{SAA}\right)_{\mu\nu} = -2\,d^{ijk}B_0F^2\frac{(p_1)_\mu}{p_1^2}$$

$$p_1^\mu\left(\Pi^{ijk}_{SVV}\right)_{\mu\nu} = 0$$

$$p_2^\nu\left(\Pi^{ijk}_{SVV}\right)_{\mu\nu} = 0$$



Dai *et.al.*, PRD99 (2019) 114015

SAA

- P and Q are the Lorentz structure of momentum, they vanish by timing $p_{1\mu}$ and $p_{2\nu}$.

$$\left(\Pi^{ijk}_{SAA}\right)_{\mu\nu} = d^{ijk} B_0 \left[-2\,F^2 \frac{(p_1)_\mu (p_2)_\nu}{p_1^2 p_2^2} + \mathcal{F}_A\left(p_1^2, p_2^2, q^2\right)\, P_{\mu\nu} + \mathcal{G}_A\left(p_1^2, p_2^2, q^2\right)\, Q_{\mu\nu}\right]$$

$$P_{\mu\nu} = \left(p_2\right)_\mu \left(p_1\right)_\nu - p_1 \cdot p_2\, g_{\mu\nu}\,,$$

$$Q_{\mu\nu} = p_1^2 \left(p_2\right)_\mu \left(p_2\right)_\nu + p_2^2 \left(p_1\right)_\mu \left(p_1\right)_\nu - p_1 \cdot p_2 \left(p_1\right)_\mu \left(p_2\right)_\nu - p_1^2\, p_2^2\, g_{\mu\nu}$$

$$\lim_{\lambda\to\infty} \left(\Pi^{ijk}_{SAA}\right)_{\mu\nu} (\lambda p_1, \lambda p_2) = -2\, d^{ijk}\, B_0 F^2 \frac{1}{\lambda^2} \frac{1}{p_1^2\, p_2^2\, q^2} \left[q^2 \left(p_1\right)_\mu \left(p_2\right)_\nu + Q_{\mu\nu} - p_1 \cdot p_2\, P_{\mu\nu}\right] + \mathcal{O}\left(\frac{1}{\lambda^3}\right)$$

$$\lim_{\lambda\to\infty} \left(\Pi^{ijk}_{SAA}\right)_{\mu\nu} (\lambda p_1, p_2) = -2\, d^{ijk}\, B_0 F^2 \frac{1}{\lambda} \frac{\left(p_1\right)_\mu \left(p_2\right)_\nu}{p_1^2 p_2^2} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)$$

$$\lim_{\lambda\to\infty} \left(\Pi^{ijk}_{SAA}\right)_{\mu\nu} (p_1, \lambda p_2) = -2\, d^{ijk}\, B_0 F^2 \frac{1}{\lambda} \frac{\left(p_1\right)_\mu \left(p_2\right)_\nu}{p_1^2 p_2^2} + \mathcal{O}\left(\frac{1}{\lambda^2}\right)$$

$$\lim_{\lambda\to\infty} \left(\Pi^{ijk}_{SAA}\right)_{\mu\nu} (\lambda p_1, q - \lambda p_1) = \mathcal{O}\left(\frac{1}{\lambda^2}\right)$$

SAA matching

- Constrains

$$\hat{L}_5 = \hat{C}_{12} = \hat{C}_{80} = \hat{C}_{85} = 0,$$
$$\lambda_6^A = \lambda_{16}^A = \lambda_{12}^S = \lambda_{16}^S = 0,$$
$$\lambda_6^{AA} = -\frac{F^2}{16 F_A^2},$$
$$\lambda_1^{SA} = \frac{1}{\sqrt{2} F_A}\left(c_d - \frac{F^2}{8 c_m}\right),$$
$$\lambda_2^{SA} = -\frac{c_d}{2\sqrt{2} F_A}.$$

- 15 couplings, 4 of them remain λ_{17}^A λ_{17}^S λ_{18}^S λ^{SAA}
- Also from $\Pi_{SS-PP}^{ij}(t)$ $F_S^{ij}(t)$, one can knows three more couplings, only 1 remain

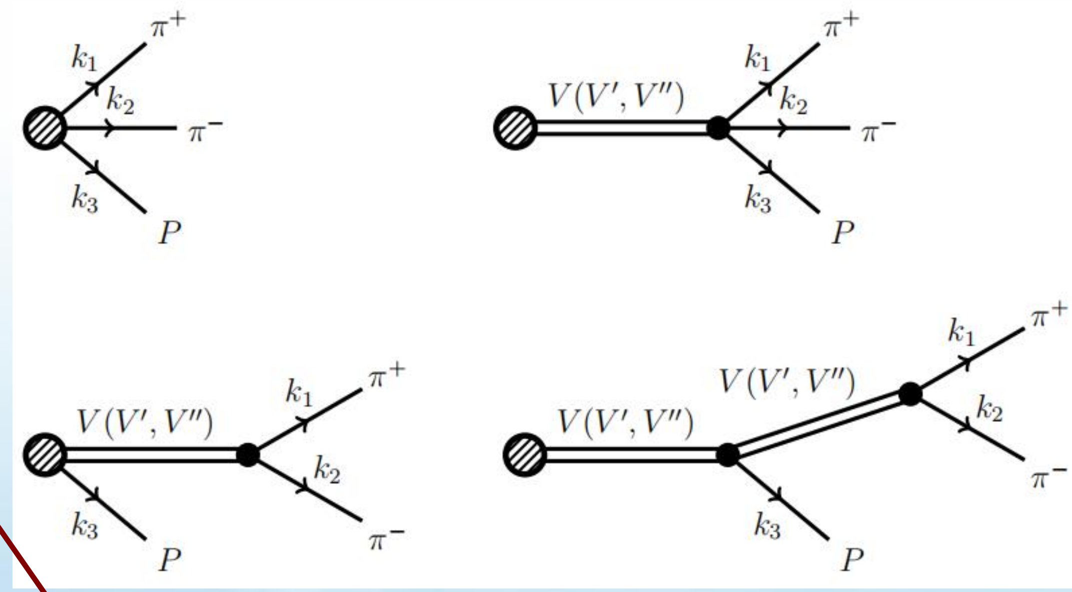
$$\lambda_{17}^S = \lambda_{18}^S = 0,$$
$$\lambda_{17}^A = 0,$$

V. Cirigliano, et.al., NPB753 (2006) 179
G. Ecker, PLB223 (1989) 425

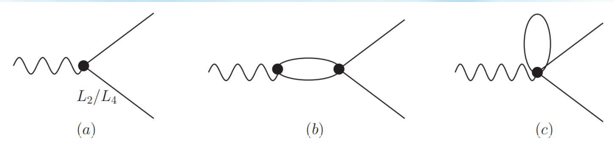
Building amplitudes

- RChT in the resonance region, excited states?

  - V', V'' has the same topologies as the ground states
  - ππ-KK FSI part by matching with Omens functions and ChPT



Guerrero, et.al., PLB 412 (1997) 382



Wang, Fang, Dai, JHEP07 (2023) 037

$$\frac{1}{M_V^2 - x} \to \frac{1}{M_V^2 - x} + \frac{\beta'_\pi}{M_{V'}^2 - x} + \frac{\beta''_\pi}{M_{V''}^2 - x}$$

Dai, et.al., PRD88 (2013) 056001

Building amplitudes

- We give a combined analysis on several channels:

$$\pi^+\pi^-, K^+K^-, \pi^+\pi^-\pi^0, \pi^+\pi^-\eta \;\; \pi^0\gamma \text{ and } \eta\gamma.$$

  - ρ-ω mixing, origined from Gasser&Leutwyler's

$$F_V^\pi = \left(1 + \frac{F_V G_V}{F^2} Q^2 \left(BW(M_\rho, \Gamma_{\rho,}, Q^2) + \beta'_{\pi\pi} BW(M_{\rho'}, \Gamma_{\rho',}, Q^2) + \beta''_{\pi\pi} BW(M_{\rho''}, \Gamma_{\rho'',}, Q^2)\right)\right.$$
$$\left(\frac{1}{\sqrt{3}} \sin\theta_V \sin\delta^\rho + \cos\delta\right)\cos\delta$$
$$- \frac{F_V G_V}{F^2} Q^2 \left(BW(M_\omega, \Gamma_{\omega,}, Q^2) + \beta'_{\pi\pi} BW(M_{\omega'}, \Gamma_{\omega',}, Q^2)\right.$$
$$\left.\left. + \beta'''_{\pi\pi} BW(M_{\omega''}, \Gamma_{\omega'',}, Q^2)\right)\left(\frac{1}{\sqrt{3}} \sin\theta_V \cos\delta - \sin\delta^\omega\right)\sin\delta^\omega\right)$$
$$\exp\left[\frac{-s}{96\pi^2 F^2}\left(\mathrm{Re}\left[A[m_\pi, M_\rho, Q^2] + \frac{1}{2} A[m_K, M_\rho, Q^2]\right]\right)\right]$$

- Not much freedom for Fit

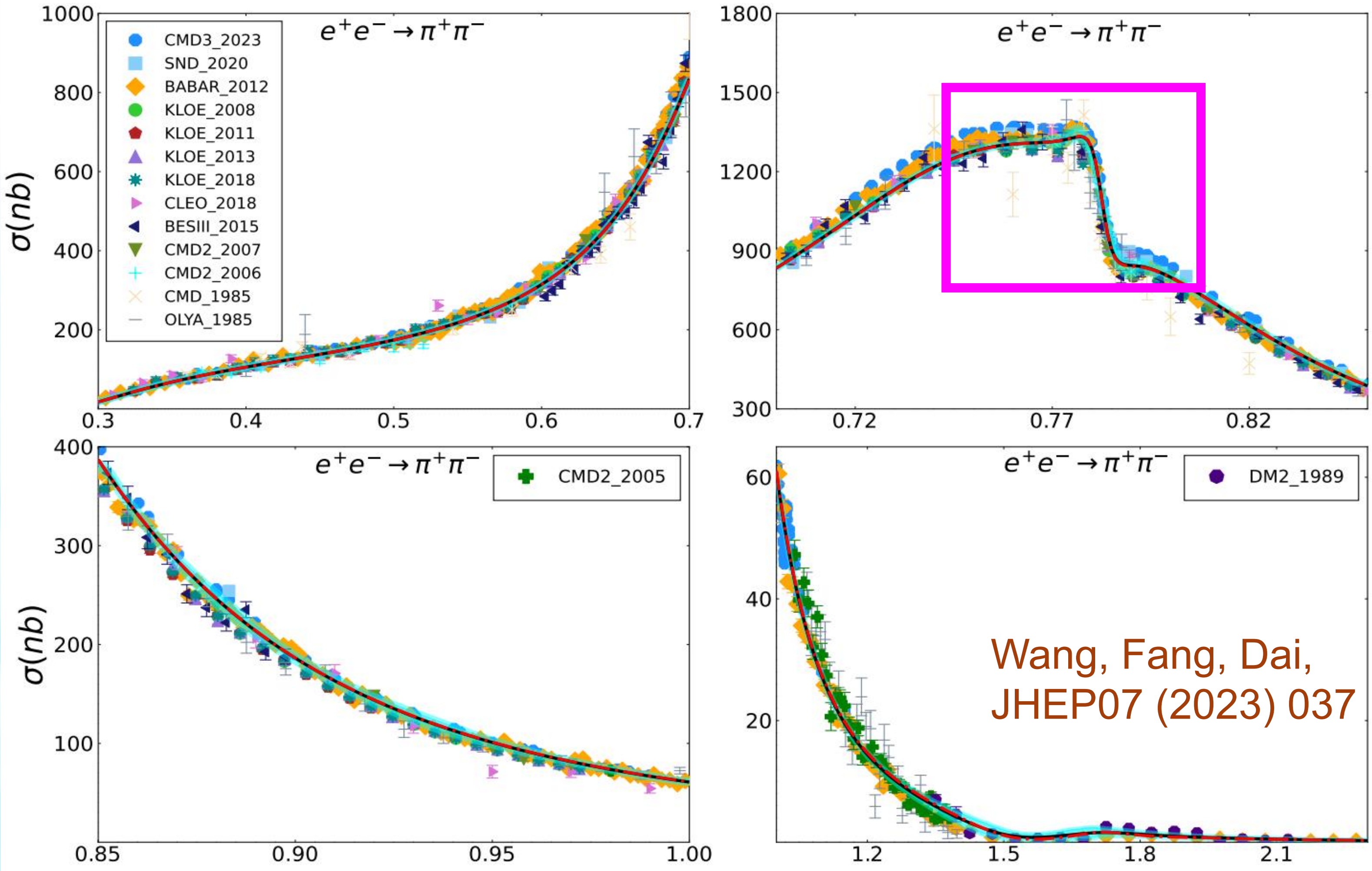
It is 1, from QCD as well as disersion relation constraints

Gasser&Leutwyler, Phys.Rept.87 (1982) 77

Guerrero&Pich, PLB 412 (1997) 382

ππ

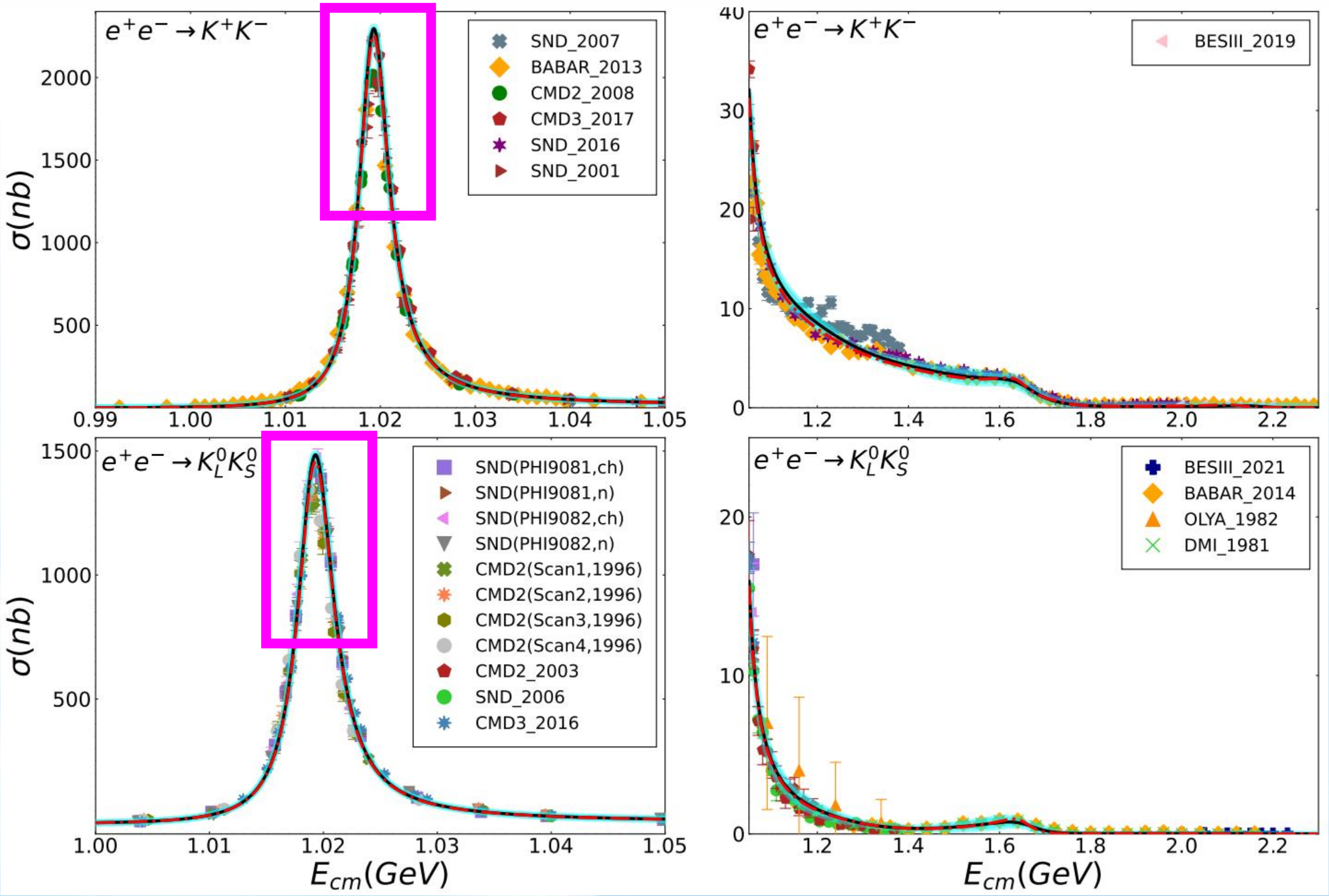
- ππ: Now closer to KLOE and BESIII's
- Latest exp: CMD-3, large discrepancy



Wang, Fang, Dai, JHEP07 (2023) 037

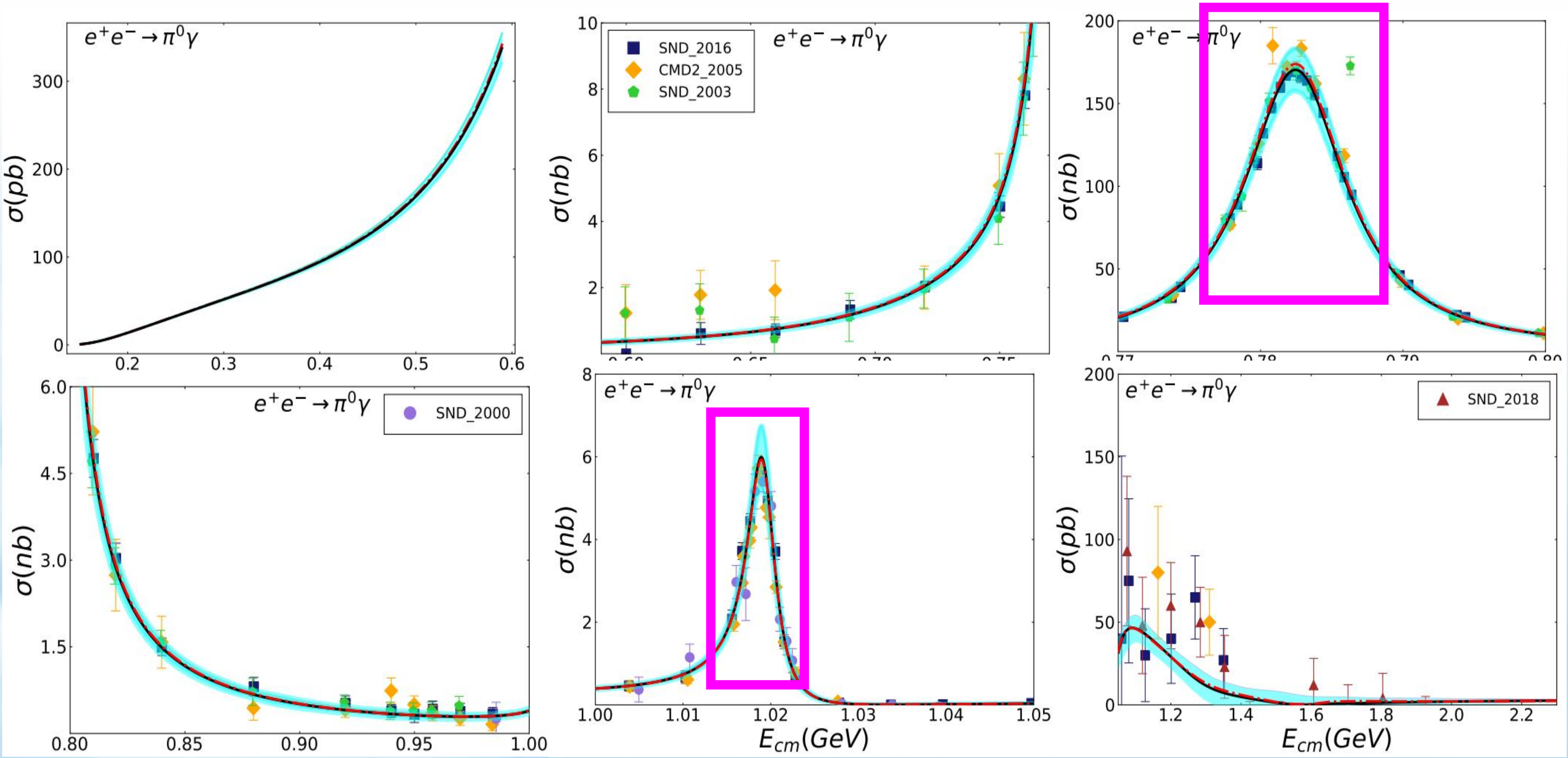
KK

- KK: data in the ϕ 'peak' have large discrepancy
- K_LK_S: further direct constraints on $\pi\pi$, KK channels



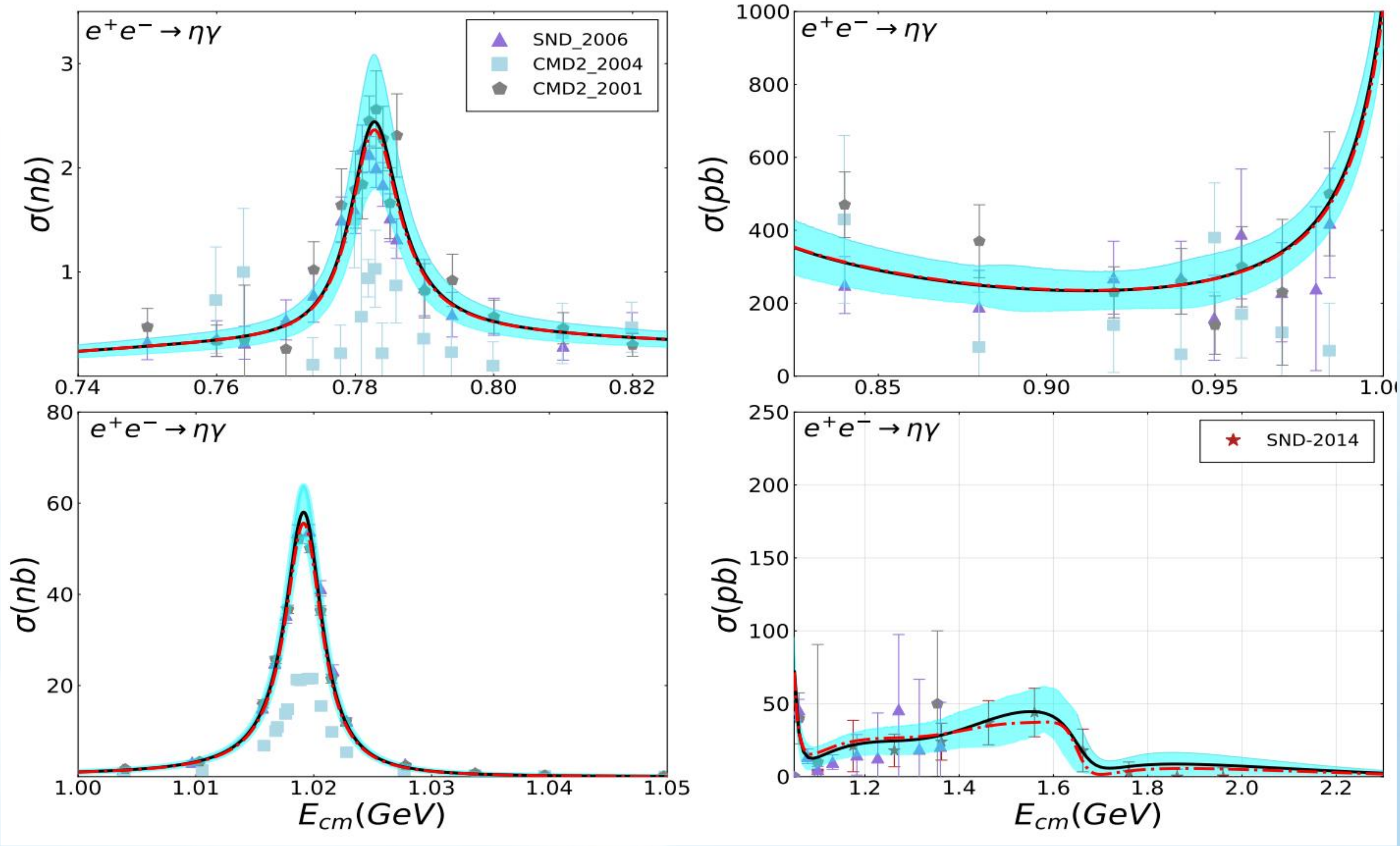
πγ

- $\pi\gamma$: helps to constrain $\pi\pi$, KK channels: ρ, ω, ϕ



ηγ

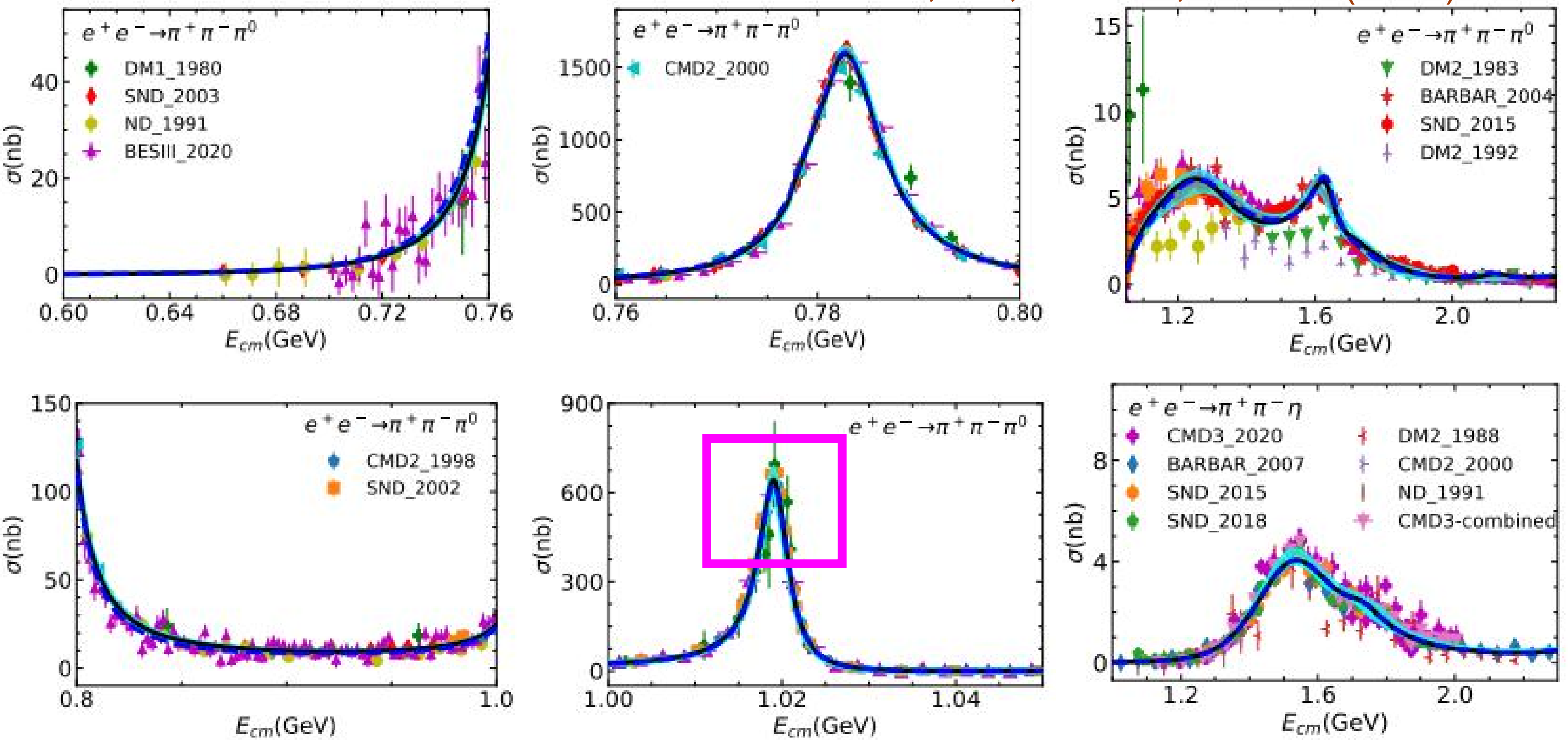
- ηγ: helps to constrain KK, and parameters of ρ, ω, ϕ



ππππ, ππη

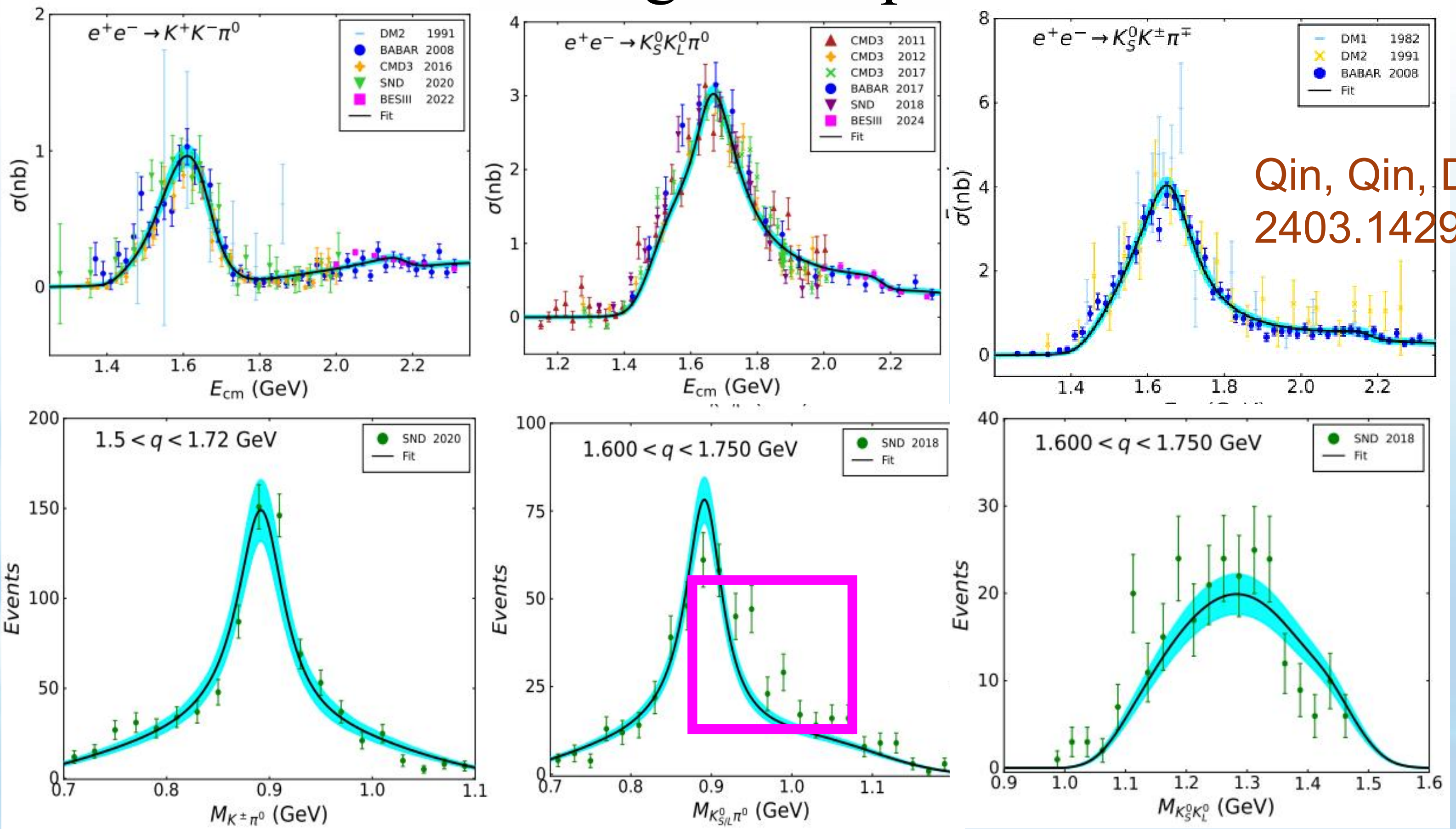
- πππ: needs more precise data in the ω ϕ region
- ππη: check our model

Qin, Dai, Portoles, JHEP03(2021)092



KKπ

- KKπ: angular distributions are helpful to constrain amplitudes
- Three body rescattering can improve it



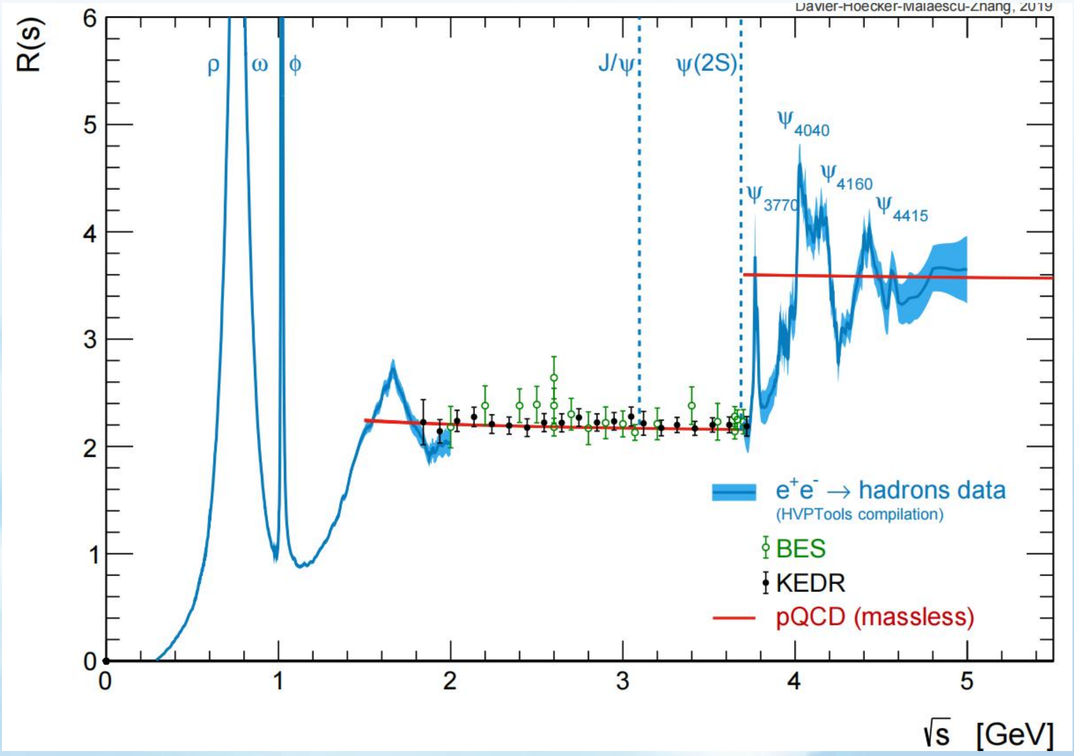
Qin, Qin, Dai, arxiv: 2403.14294 [hep-ph]

R value

- Cross sections needs to be corrected

$$R_{\mathrm{h}}(s) = \frac{3s}{4\pi\alpha_e^2(s)}\sigma\left(e^+e^- \to \text{ hadrons }\right) \qquad \mathrm{Re}\Pi_{\mathrm{had}}(s) = -\frac{\alpha_e(0)s}{3\pi}\mathrm{P}\int_{s_{\mathrm{th}}}^{\infty}\frac{R(s')}{s'(s'-s)}ds'$$

- R values are input from PDG



Davier *et.al.*,
EPJC 80 (2020) 3, 241

g-2: HVP-LO

- Other channels are taken from data-driven or QCD

J/ψ (BW integral)	6.28 ± 0.07
$\psi(2S)$ (BW integral)	1.57 ± 0.03
R data $[3.7 - 5.0]$ GeV	$7.29 \pm 0.05 \pm 0.30 \pm 0.00$
$R_{\rm QCD}$ $[1.8 - 3.7\ {\rm GeV}]_{uds}$	$33.45 \pm 0.28 \pm 0.65_{\rm dual}$
$R_{\rm QCD}$ $[5.0 - 9.3\ {\rm GeV}]_{udsc}$	6.86 ± 0.04
$R_{\rm QCD}$ $[9.3 - 12.0\ {\rm GeV}]_{udscb}$	1.21 ± 0.01
$R_{\rm QCD}$ $[12.0 - 40.0\ {\rm GeV}]_{udscb}$	1.64 ± 0.00
$R_{\rm QCD}$ $[> 40.0\ {\rm GeV}]_{udscb}$	0.16 ± 0.00
$R_{\rm QCD}$ $[> 40.0\ {\rm GeV}]_{t}$	0.00 ± 0.00

- HVP-LO: $694.10 \pm 3.14 \times 10^{-10}$
- Ours: $a_\mu = 11659181.1 \pm 3.5 \times 10^{-11}$

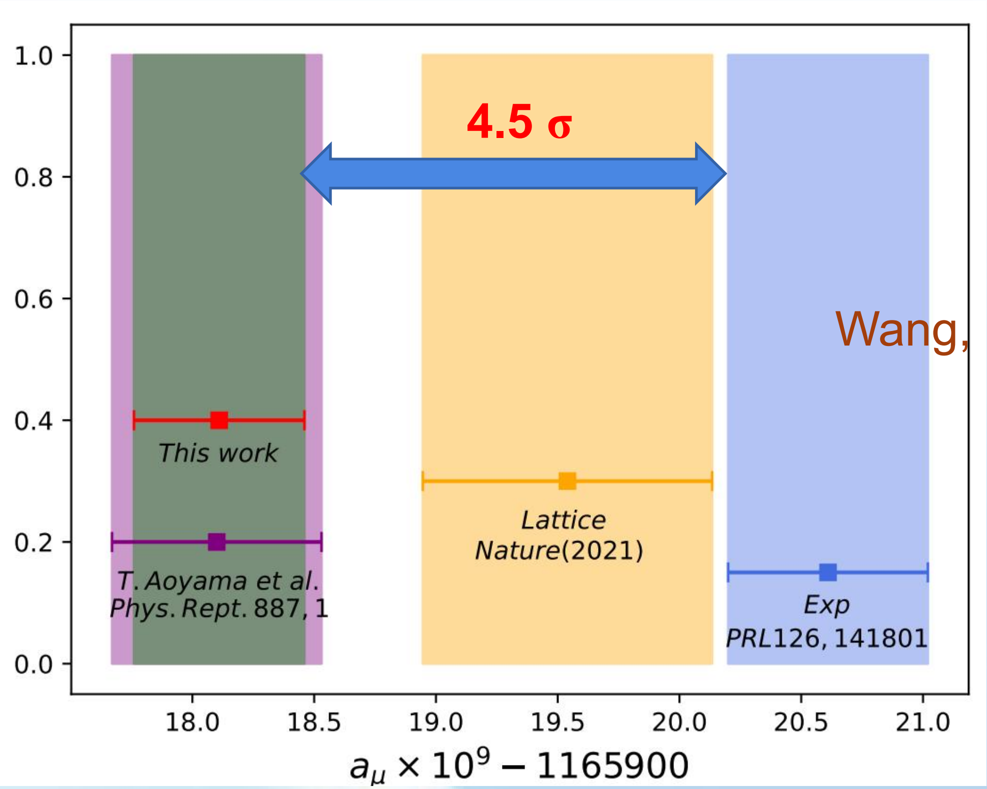
$708.7(5.3) \times 10^{-10}$

Nature 593 (2021) 7857, 51-55

HVP

- Ours: $a_\mu = 11659181.1 \pm 3.5 \times 10^{-11}$
- It differs 4.5σ from latest experiment's
  - 3.9σ If HLBL part repleaced with latest LQCD's

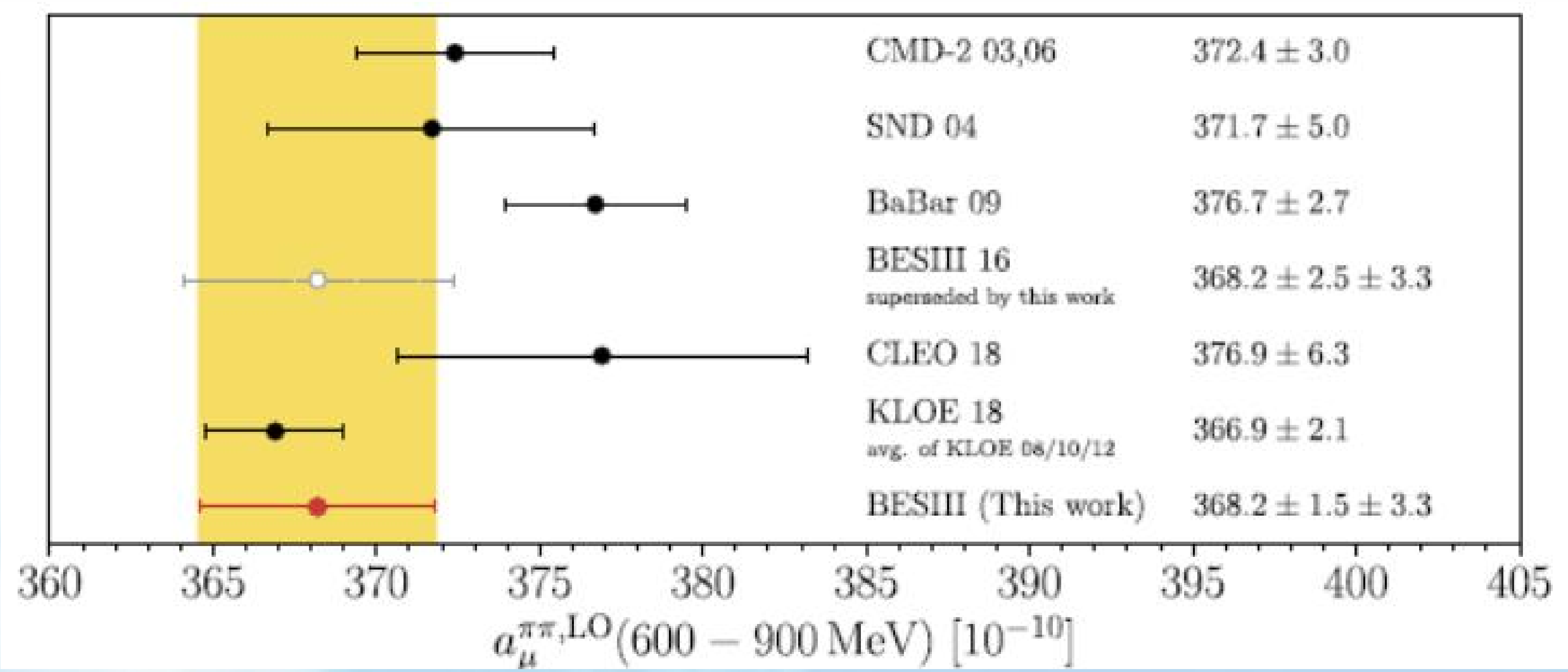
T. Blum, et.al., arxiv:2304.04423 [hep-lat]



Wang, Fang, Dai

Experiment

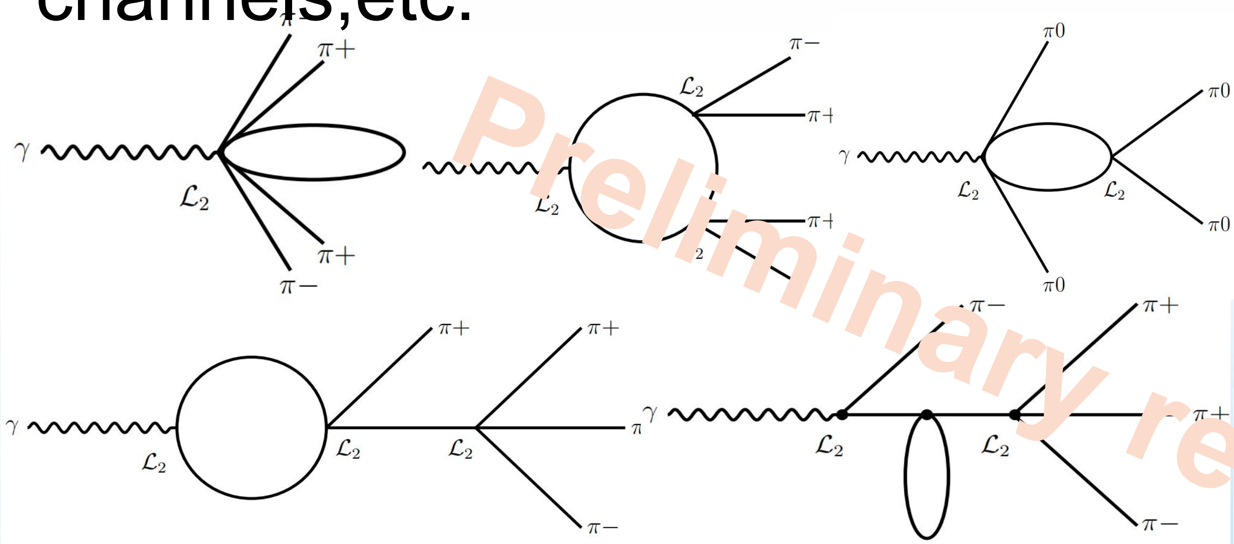
- Future experiments?

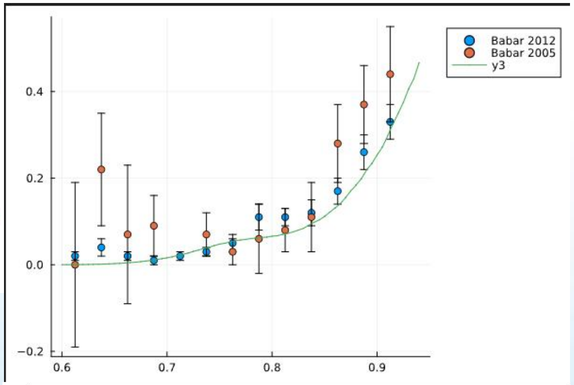


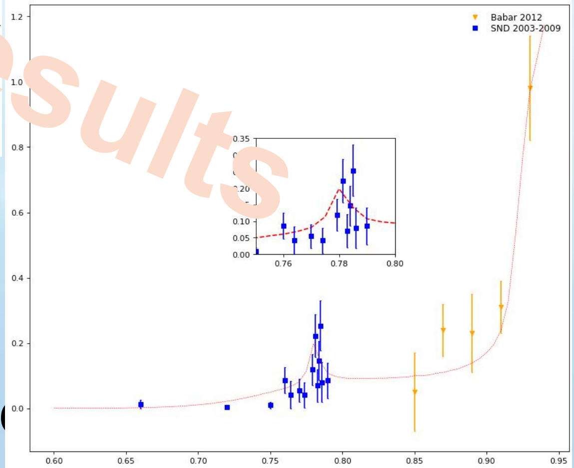
Guangshun Huang, talk at HNU

Four body final states?

Four body final states are important: ππππ, ππKK channels,etc.



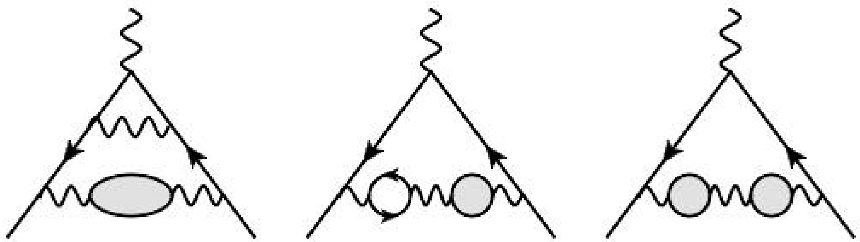


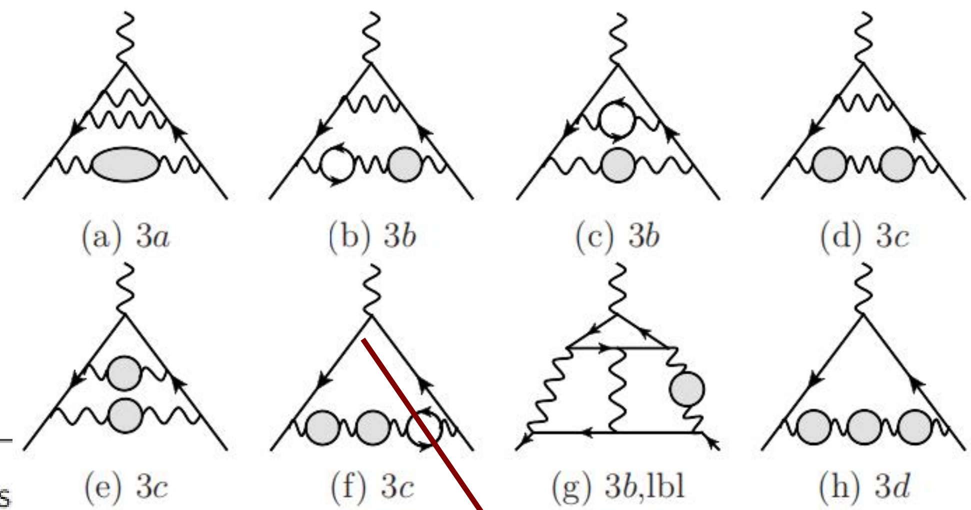


- ChPT’s << data, in resonance energy r
- FSI?
- Resonances?

HVP: NLO, NNLO?

- More channels (also high energy ones) to give a complete estimation?





$\times 10^{-12}$	$\pi\pi$	KK	$\pi\pi\pi$	$\pi\pi\eta$	S	(PLB 734,144)
2a	-1369±8	-79.8±2.8	-145±3	-5.93±0.46	-1600±9	-2090
2b	776±5	37.6±1.3	74.7±1.8	2.37±0.18	891±5	1068
2c		22.4±0.2			22.4±0.2	35
a_μ^{NLO}					-687±10	−987±9
3a	45.4±0.3	3.11±0.11	5.20±0.12	0.267±0.021	54.0±0.3	80
3b	-24.8±0.2	-1.62±0.06	-2.78±0.06	-0.131±0.010	-29.3±0.2	-41
3bLBL	58.0±0.3	3.47±0.12	6.19±0.14	0.268±0.021	67.9±0.4	91
3c		-2.34±0.02			-2.34±0.02	-6
3d		0.0249±0.0004			0.0249±0.0004	0.05
a_μ^{NNLO}					90.3±0.5	124±1

Kurz, et.al.
PLB 734 (2014) 144

Refine our results by considering other channels of three, four body final states .

4、Summary

FSI

Amplitude analsysis connects QFT principles and Exp. FSI needs to be considered when performing amplitude analysis.

RChT

RChT+FSI are powerful to work in the intermediate energy region, between ChPT and QCD.

HVP

Our g-2 has a significant discrepancy with the latest FNAL's. Processes of multi-body channels needs to be studied. $\pi\pi\pi\pi$, $\pi\pi KK$?

Next?

Further study of light hadrons is neccessary to give a more reliable answer to muon g-2; Discrepancy between LQCD v.s. data driven; Improving ChEFT+FSI?

Thank You For your patience !