YANG MODEL REVISITED

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SUMMARY

- Noncommutative geometry in curved spacetime
- Snyder model
- Yang model
- Yang-Poisson model and its realizations
- Hopf algebra of the Yang model
- Applications

Quantum gravity and noncommutative geometry

- At present, no complete theory of quantum gravity is available
- However, it is known that the predictions of quantum mechanics and general relativity imply the existence of a minimal measurable length of the scale of Planck length $L_P = \sqrt{\frac{\hbar G}{c^3}} = 10^{-33}$ cm
- Therefore, the properties of spacetime at this scale must be rather different from the usual ones.
- Among the proposals for a model of spacetime at these scales, noncommutative geometry has a relevant role.
- Noncommutative geometry is based on the assumption that the components of the position operator do not commute, leading to the impossibility of localizing a particle exactly
- Among various approaches to this field. an important role is played by Hopf algebra formalism

Noncommutative geometry in curved spacetime

• Noncommutative geometry is usually defined on flat spacetime

• Noncommutative geometry in curved spacetime has earned some interest recently because of possible implications for astrophysical observations, like the possible time delay of photons from distant sources

• However, also its formal aspects are noticeable, in particular the relations between curvature of spacetime and of momentum space

• Moreover, these models relate spacetime at microscopic and macroscopic scales

• A model of this kind was proposed by C.N. Yang already in 1947 (Yang, PRD 1947)

• We review this framework and discuss some recent progress and generalizations

The Snyder algebra

• In 1947 Snyder proposed the first model of noncommutative geometry. (Snyder, PRD 1947)
• His aim was to define a theory that included a fundamental length without breaking the Lorentz invariance
• This was realized by deforming the commutation relations of the Heisenberg algebra
• The model was defined through an algebra that besides the deformed Heisenbeg algebra, generated by positions $\hat{x}_\mu$ and momenta $\hat{p}_\mu$, contained the Lorentz algebra with generators $J_{\mu\nu}$

$$[\hat{x}_\mu, \hat{x}_\nu] = i\beta J_{\mu\nu}, \quad [\hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{x}_\mu, \hat{p}_\mu] = i(\eta_{\mu\nu} + \beta \hat{p}_\mu \hat{p}_\nu),$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\big(\eta_{\mu\rho} J_{\nu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} + \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\nu\sigma} J_{\mu\rho}\big),$$

$$[J_{\mu\nu}, \hat{p}_\lambda] = i\left(\eta_{\mu\lambda} \hat{p}_\nu - \eta_{\lambda\nu} \hat{p}_\mu\right), \quad [J_{\mu\nu}, \hat{x}_\lambda] = i\left(\eta_{\mu\lambda} \hat{x}_\nu - \eta_{\nu\lambda} \hat{x}_\mu\right)$$

• In particular, the $\hat{x}_\mu$ components do not commute among themselves

• The coupling constant β has dimension of inverse mass square and may be identified with $1/M^2_{Planck}$

• In contrast with the most common models of noncommutative geometry, the commutators are functions of the phase space variables: this allows them to be compatible with a linear action of the Lorentz symmetry, so that the Poincaré algebra is not deformed. However, translations (generated by the p_μ) act in a nontrivial way on position variables

• The Snyder model can be interpreted as describing flat spacetime with a curved momentum space

• In fact, the subalgebra generated by $J_{\mu\nu}$ and $\hat{x}_\mu$ is isomorphic to the de Sitter algebra $so(1,4)$, and the Snyder momentum space has the same geometry as de Sitter spacetime

The Yang algebra

• Soon after Snyder, Yang proposed a generalization of the model where also the momentum variables do not commute, like in de Sitter spacetime (Yang, PRD 1947)

• The algebra has the form of a $so(1,5)$ algebra, with 15 generators

$$[\hat{x}_\mu, \hat{x}_\nu] = i\beta J_{\mu\nu}, \quad [\hat{p}_\mu, \hat{p}_\nu] = i\alpha J_{\mu\nu}, \quad [\hat{x}_\mu, \hat{p}_\nu] = i\eta_{\mu\nu} h,$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\big(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} + \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\nu\sigma}J_{\mu\rho}\big),$$

$$[J_{\mu\nu}, \hat{p}_\lambda] = i\left(\eta_{\mu\lambda}\hat{p}_\nu - \eta_{\lambda\nu}\hat{p}_\mu\right), \quad [J_{\mu\nu}, \hat{x}_\lambda] = i\left(\eta_{\mu\lambda}\hat{x}_\nu - \eta_{\nu\lambda}\hat{x}_\mu\right)$$

$$[h, \hat{x}_\mu] = i\beta\hat{p}_\mu, \quad [h, \hat{p}_\mu] = -i\alpha\hat{x}_\mu, \quad [J_{\mu\nu}, h] = 0$$

• α has dimension of inverse length square and may be identified with the cosmological constant, while β is the same as in the Snyder model

• The Yang algebra contains as subalgebras both the de Sitter and the Snyder algebras, and therefore describes a noncommutative model in a spacetime of constant curvature

• In order to close the algebra, Yang had to introduce a new generator h which rotates positions into momenta, but whose physical interpretation is not evident

• The previous algebra is invariant under a generalized Born duality (Born, RMP 1949)

$$\alpha \leftrightarrow \beta, \quad \hat{x}_\mu \to -\hat{p}_\mu, \quad \hat{p}_\mu \to \hat{x}_\mu, \quad J_{\mu\nu} \leftrightarrow J_{\mu\nu}, \quad h \leftrightarrow h$$

• The isomorphism with the $so(1,5)$ algebra can be obtained by identifying

$$M_{\mu\nu} = J_{\mu\nu}, \qquad M_{\mu 4} = \hat{x}_\mu, \qquad M_{\mu 5} = \hat{p}_\mu, \qquad M_{45} = h$$

where M_{AB} $(A, B = 0, \ldots, 5)$ are the generators of $so(1,5)$

Triply special relativity

• There exists a different generalization of the Snyder algebra on curved space, known as triply special relativity that does not include h, but is nonlinear. (Kowalski, Smolin, PRD 2004)

• In particular, in that case the deformed Heisenberg subalgebra takes the form

$$[\hat{x}_\mu, \hat{x}_\nu] = i\beta J_{\mu\nu}, \quad [\hat{p}_\mu, \hat{p}_\nu] = i\alpha J_{\mu\nu}$$

$$[\hat{x}_\mu, \hat{p}_\nu] = i(\eta_{\mu\nu} + \alpha\hat{x}_\mu\hat{x}_\nu + \beta\hat{p}_\mu\hat{p}_\nu + \sqrt{\alpha\beta}(\hat{x}_\mu\hat{p}_\nu + \hat{p}_\mu\hat{x}_\nu - J_{\mu\nu}))$$

• In this case, one can interpreted the phase space as a coset space

$$\frac{so(1,5)}{so(1,3) \times so(2)}$$

Two interpretations of the Yang algebra are possible:

• Take the model as it is, with its 15 generators. This allows one to construct the Hopf algebra structure, with the related star product, etc.

◦ However, in this case one has to consider an extended phase space and the interpretation of the new degrees of freedom is not obvious

• Take a nonlinear realization on canonical phase space spanned by x_μ and p_μ, with $J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\nu$ and $h = h(x, p)$

◦ In this case the interpretation is easier and one can include the Yang model in the same family of nonlinear realizations as TSR, identifying the phase space with a coset space.

◦ However, one can no longer define star products etc.

Yang-Poisson model

• We start following the second route, and discuss the classical limit of the Yang model, in which commutators are replaced by Poisson brackets

• This is much easier because of the absence of ordering problems

• We have (Meljanac, SM, IJMPA 2023)

$$\{\hat{x}_\mu, \hat{x}_\nu\} = \beta J_{\mu\nu}, \quad \{\hat{p}_\mu, \hat{p}_\nu\} = \alpha J_{\mu\nu}, \quad \{\hat{x}_\mu, \hat{p}_\nu\} = \eta_{\mu\nu} h,$$

$$\{h, \hat{x}_\mu\} = \beta \hat{p}_\mu, \quad \{h, \hat{p}_\mu\} = -\alpha \hat{x}_\mu$$

• We look for an expression of $h(x, p)$ that satisfies the previous Poisson brackets

• We make the ansatz

$$\hat{x}_\mu = f(p^2, z)\, x_\mu, \qquad \hat{p}_\mu = g(x^2, z)\, p_\mu$$

$$h = h(x^2, p^2, z)$$

where $z = x \cdot p$ and f and g are functions to be determined.

Realization of the Yang-Poisson model

• The only nontrivial brackets to be checked are those of the deformed Heisenberg algebra, which give rise to partial differential equations. The x–x and p–p brackets have solutions

$$f = \sqrt{1 - \beta p^2 + \phi_1(z)}, \qquad g = \sqrt{1 - \alpha x^2 + \phi_2(z)}$$

with arbitrary functions ϕ_1 an ϕ_2, while the x–p brackets give

$$\phi_1\phi_2 + \phi_1 + \phi_2 = \alpha\beta z^2, \qquad h = fg$$

with solution depending on one parameter c

$$\phi_1(z) = \frac{\sqrt{1 + 4c(1-c)z^2} - 1}{2(1-c)}, \quad \phi_2(z) = \frac{\sqrt{1 + 4c(1-c)z^2} - 1}{2c}$$

Then,

$$\hat{x}_\mu = \sqrt{1 - \beta p^2 + \phi_1(z)}\; x_\mu, \quad \hat{p}_\mu = \sqrt{1 - \alpha x^2 + \phi_2(z)}\; p_\mu$$

and

$$h = \sqrt{[1 - \beta p^2 + \phi_1(z)]\,[1 - \alpha x^2 + \phi_2(z)]}$$

• In terms of the original variables,

$$h=\sqrt{1-\alpha\hat{x}^2-\beta\hat{p}^2-\alpha\beta\frac{J^2}{2}}$$

• A particularly interesting solution is obtained by assuming symmetry under the exchange of x and p, as is natural in view of the Born duality of the model. In this case, $\phi_1=\phi_2=\phi$, i.e. $c=\frac{1}{2}$, and we obtain

$$\phi=\sqrt{1+\alpha\beta z^2}-1$$

and then

$$\hat{x}_\mu=\sqrt{\sqrt{1+\alpha\beta z^2}-\beta p^2}\ x_\mu,\quad \hat{p}_\mu=\sqrt{\sqrt{1+\alpha\beta z^2}-\alpha x^2}\ p_\mu$$

This gives an exact realization of the Yang model, symmetric for $x\leftrightarrow p$ and $\alpha\leftrightarrow\beta$.

Realizations of the quantum Yang model

• In the quantum case, finding a realization is more difficult, and can only be achieved by a perturbative calculation in the coupling parameters α and β The simplest case is (Meljanac et al., JMP 2023)

$$\hat{x}_\mu = x_\mu - \frac{\beta^2}{4} x_\mu p^2 - \frac{\beta^4}{16} x_\mu p^4 + \frac{\alpha^2\beta^2}{8} x_\mu \, x{\cdot}p \; p{\cdot}x \, + \mathrm{h.c.}$$

$$\hat{p}_\mu = p_\mu - \frac{\alpha^2}{4} p_\mu x^2 - \frac{\alpha^4}{16} p_\mu x^4 + \frac{\alpha^2\beta^2}{8} p_\mu \, p{\cdot}x \; x{\cdot}p \, + \mathrm{h.c.}$$

with

$$h = 1 - \frac{1}{2}\left(\alpha^2 x^2 + \beta^2 p^2\right) - \frac{1}{8}\left(\alpha^2 x^2 - \beta^2 p^2\right)^2 + \frac{\alpha^2\beta^2}{2} \, x{\cdot}p \; p{\cdot}x$$

• However, at leading order in $\hbar$, one gets the classical result

Star product for the Yang algebra

• The most useful framework for noncommutative geometry is that of Hopf algebras
• It is possible to apply this formalism also to the Yang model, provided one takes all its generators M_{AB} as primary variables.
• We shall not go into details. We only recall that due to noncommutativity, the addition law of momenta is deformed.
• The deformation can be expressed by means of a star product. In our case, for plane waves, one has

$$e^{\frac{i}{2}s^{AB}M_{AB}} \star e^{\frac{i}{2}t^{CD}M_{CD}} = e^{\frac{i}{2}\mathcal{D}^{AB}(s,t)M_{AB}}$$

where s_{AB} and t_{AB} are antisymmetric tensors that describe the "momenta" conjugated to the primary variables M_{AB} and $\mathcal{D}^{AB}$ encodes the deformed addition law

• It may be useful to explicitly write down the four-dimensional expression of $\mathcal{D}^{AB}(s,t)$:
setting $\mathcal{D}^{\mu} = \mathcal{D}^{\mu 4}$, $\bar{\mathcal{D}}^{\mu} = \mathcal{D}^{\mu 5}$, $\mathcal{D} = \mathcal{D}^{45}$, one has

$$\mathcal{D}^{\mu\nu}(s,t) = s^{\mu\nu} + t^{\mu\nu} - \frac{1}{2}\Big(s^{\mu\lambda}t^{\nu}{}_{\lambda} + \beta s^{\mu}t^{\nu} + \alpha\bar{s}^{\mu}\bar{t}^{\nu} + \gamma(s^{\mu}\bar{t}^{\nu} + \bar{s}^{\mu}t^{\nu}) - (\mu \leftrightarrow \nu)\Big)$$

$$\mathcal{D}^{\mu}(s,t) = s^{\mu} + t^{\mu} - \frac{1}{2}(s^{\mu\lambda}t_{\lambda} - t^{\mu\lambda}s_{\lambda} + \gamma(s^{\mu}t - st^{\mu}) + \alpha(\bar{s}^{\mu}t - s\bar{t}^{\mu}))$$

$$\bar{\mathcal{D}}^{\mu}(s,t) = \bar{s}^{\mu} + \bar{t}^{\mu} - \frac{1}{2}(s^{\mu\lambda}\bar{t}_{\lambda} - \bar{s}_{\lambda}t^{\mu\lambda} - \gamma(\bar{s}^{\mu}t - s\bar{t}^{\mu}) + \beta(s^{\mu}t - st^{\mu}))$$

$$\mathcal{D}(s,t) = s + t - \frac{1}{2}(s^{\lambda}\bar{t}_{\lambda} - \bar{s}^{\lambda}t_{\lambda})$$

where $\gamma = \sqrt{\alpha\beta}$ and $s^{\mu\nu}$, $s^{\mu} = s^{\mu 4}$, $\bar{s}^{\mu} = s^{\mu 5}$, $s = s^{45}$ are the 4D components of s^{AB}, conjugated to $J_{\mu\nu}$, x_{μ}, p_{μ} and h, resp.

• Clearly, the physical interpretation of these "momenta" is not obvious

Generalizations

• It is possible to generalize the Yang algebra by including κ-deformations of both position and momentum space with parameters a_μ and b_μ (Lukierski et al, arxiv 2023)

$$
\begin{aligned}
[x_\mu, x_\nu] &= i\,(\beta J_{\mu\nu} + a_\mu x_\nu - a_\nu x_\mu)\,, \\
[p_\mu, p_\nu] &= i\,(\alpha J_{\mu\nu} + b_\mu p_\nu - b_\mu p_\nu)\,, \\
[x_\mu, p_\nu] &= i\,(\eta_{\mu\nu} h + b_\mu x_\nu - a_\nu p_\mu + \gamma J_{\mu\nu})\,, \\
[J_{\mu\nu}, x_\lambda] &= i\,(\eta_{\mu\lambda} x_\nu - \eta_{\nu\lambda} x_\mu + a_\mu J_{\lambda\nu} - a_\nu J_{\lambda\mu})\,, \\
[J_{\mu\nu}, p_\lambda] &= i\,(\eta_{\mu\lambda} p_\nu - \eta_{\nu\lambda} p_\mu + b_\mu J_{\lambda\nu} - b_\nu J_{\lambda\mu})\,, \\
[J_{\mu\nu}, h] &= i\,(b_\nu x_\mu - b_\mu x_\nu - a_\nu p_\mu + a_\mu p_\nu)\,, \\
[h, x_\mu] &= i\,(\beta p_\mu - \gamma x_\mu - a_\mu h)\,, \\
[h, p_\nu] &= i\,(-\alpha x_\mu + \gamma p_\mu + b_\mu h)
\end{aligned}
$$

• This algebra is still isomorphic to $so(1,5)$, but now the action of the Lorentz invariance is deformed

Applications

• A physical consequence of Yang model is a deformation of the Heisenberg uncertainty relations. In fact, in 3D

$$\Delta x_i \Delta p_j \geq \frac{1}{2} \big| \langle [x_i, p_j] \rangle \big| = \frac{1}{2} \big| \langle h(x,p) \rangle \big| \delta_{ij}$$

• One can also calculate corrections to the dynamics of simple models due to the nontrivial symplectic structure, with possible applications to astrophysical observations

• Finally, a more ambitious goal would be to build a quantum field theory based on this framework

• Some of these applications require the use of an extended phase space. In this case the physical interpretation of the additional coordinates needs to be clarified