What is the physical and philosophical consequence of Hilbert's famous theorem, according to which Bolyai-Lobachevskii plane geometry cannot be implemented globally as internal geometry of a surface in 3-dimensional space?

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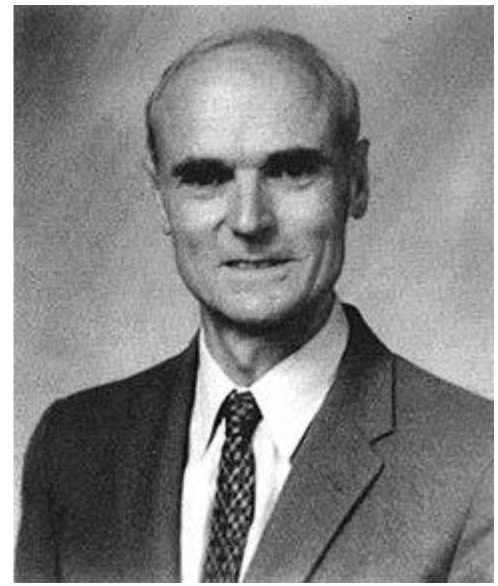
Miercurea-Ciuc

In my presentation, I would like to draw the audience's attention to two important papers in the history of science.The first is on Hilbert's very important theorem, which he published in Grundlagen der Geometrie, the second is on one of Coxeter's interesting but less cited papers.

David Hilbert

Born: 23 January 1862,Königsberg or Wehlau, Kingdom of Prussia
Died: 14 February 1943 (aged 81), Göttingen, Germany



Harold Scott MacDonald "Donald" Coxeter

Born: 9 February 1907 London, England,
Died: 31 March 2003 (aged 96), Toronto, Ontario, Canada

Hilbert published the perfect system of Euclidean geometry in 1901. As Hilbert mentioned, many people did not even understand why this monumental work was needed at the beginning of the 20th century. On the other hand, Hilbert, who is also the creator of the formalist mathematical philosophical trend, felt the need to finally clarify the system of Euclidean geometry after 2000 years. This was required by the formalist worldview itself, in order to place the mathematical structure of the world on an absolutely logically pure basis. Of course, in this monumental work, Hilbert also laid the logical foundations of Bolyai-Lobacevskii's geometry.

And he gifted the mathematical community with one more great theorem, namely his theorem that there is no complete surface with negative constant curvature in 3-dimensional space! Today, in the language of modern differential geometry, we formulate this as saying that the Bolyai-Lobachevskiy plane cannot be embedded in 3-dimensional space.What does this mean? In my opinion, this means that Bolyai-Lobacevskii geometry is not realized in 3-dimensional space! Of course, many people do not like this strong statement: because what does mathematical existence mean? However, we are working with these concepts step by step. If someone says that he has found a right-angled 3-angle in which the Pythagorean theorem is not true, we immediately think that something is wrong with the person making the statement. Well, it's similar to when someone says that some gadget works according to Bolyai-Lobacevsky geometry in 3-dimensional space. Because such a gadget does not exist!

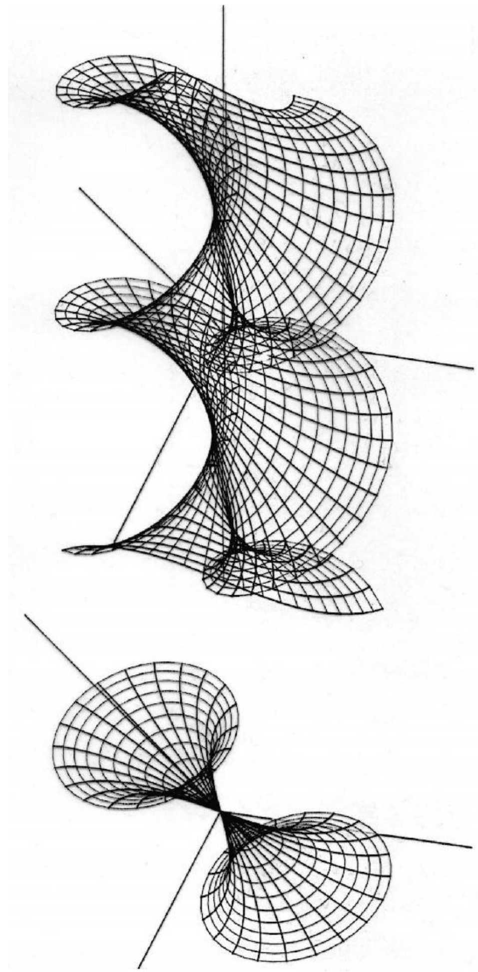
This was built, this table, the chairs on which the honored participants sit, were designed based on Euclidean geometry. But they cannot make a chair based on Bolyai-Lobachevskiy geometry. This is what Hilbert's famous theorem means.
Of course, in quantum mechanical dimensions, it might be possible there... Because in the infinite, countable dimensional, separable Euclidean space, Bolyai-Lobachevskii geometry can already be realized. (Because Ludwig Bieberbach proved this).

Ludwig Georg Elias Moses Bieberbach

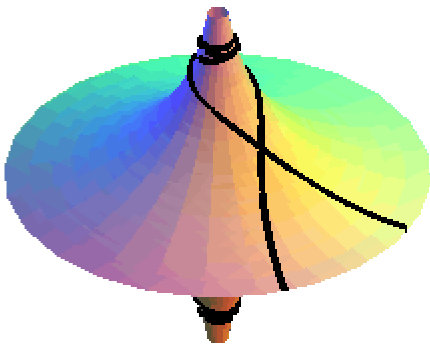
Born: 4 December 1886, Goddelau, Grand Duchy of Hesse, German Empire
Died: 1 September 1982 (aged 95), Oberaudorf, Upper Bavaria, West Germany

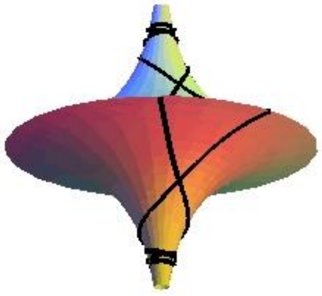
Mathematics is an exact science, its statements are true a priori, there are no exceptions. So, no matter how depressing it is, Bolyai-Lobachevskii geometry is not realized in its entirety in the 3-dimensional Euclidean space. Let me give you a not-so-good analogy: just as the Klein bottle cannot be realized in 3-dimensional space, Bolyai-Lobachevskii's planar geometry cannot be realized either.

Klein bottle (Nonorientable)



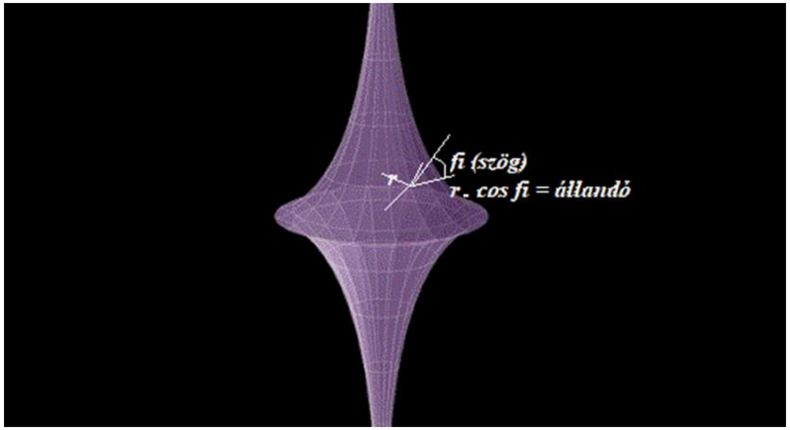
It was not by chance that Hilbert included this theorem in his Grundlagen der Geometrie. He knew, felt and understood the importance and epistemological background of this proposition.
Unfortunately, we forgot about this later! (Note that on the pseudosphere and on all surfaces with negative constant curvature, the Bolyai-Lobachevskii plane geometry is realized in a small (very small range).

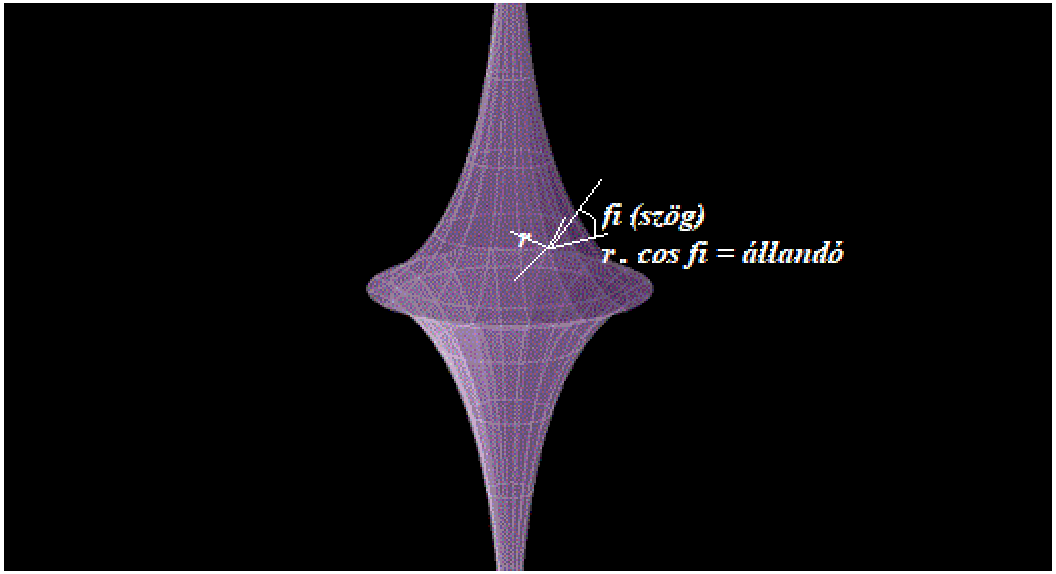




Alexis Claude Clairaut (French pronunciation: [alɛksi klod klɛʁo]; 13 May 1713 – 17 May 1765) was a French mathematician, astronomer, and geophysicist. He was a prominent Newtonian whose work helped to establish the validity of the principles and results that Sir Isaac Newton had outlined in the *Principia* of 1687. Clairaut was one of the key figures in the expedition to Lapland that helped to confirm Newton's theory for the figure of the Earth. In that context, Clairaut worked out a mathematical result now known as "Clairaut's theorem". He also tackled the gravitational three-body problem, being the first to obtain a satisfactory result for the apsidal precession of the Moon's orbit. In mathematics he is also credited with Clairaut's equation and Clairaut's relation.

According to Clairaut's theorem: the radius on the surface of rotation multiplied by the elevation angle of the geodetic will be CONSTANT! With this, we follow the path of the geodetics.

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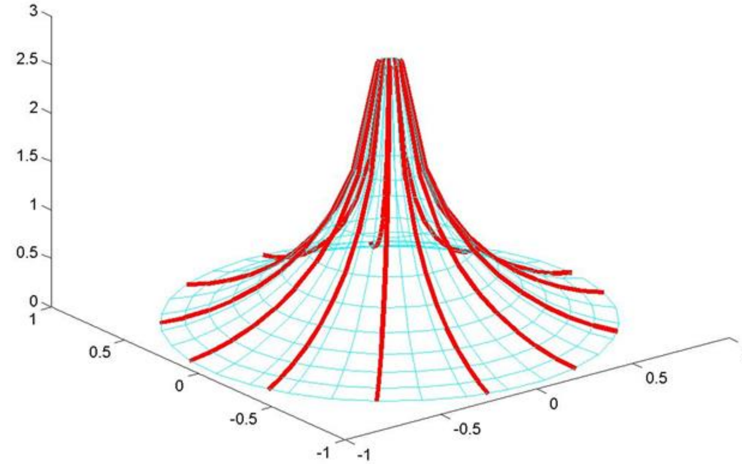

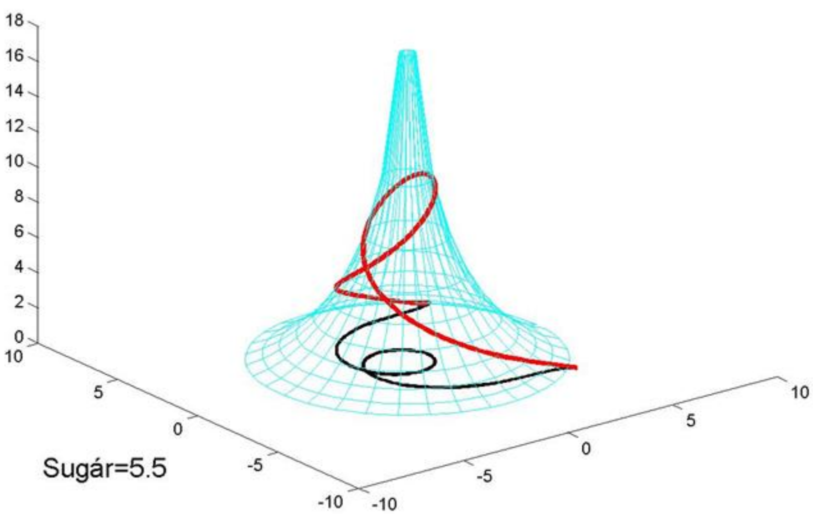
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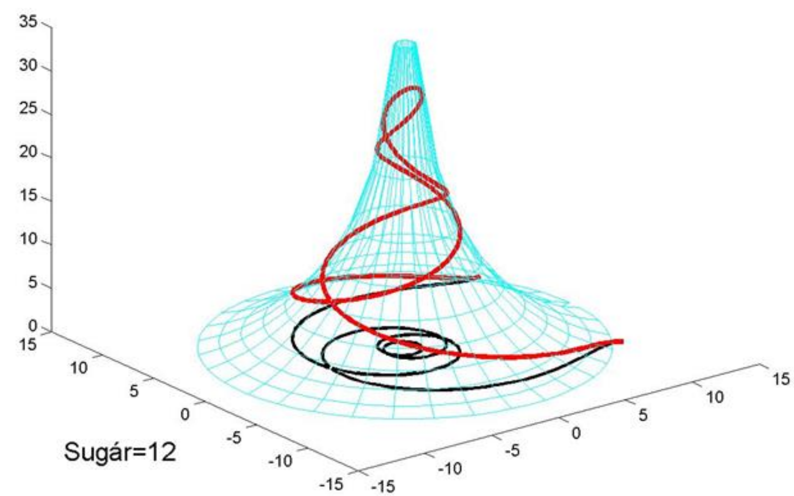
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Isn't the name of this conference Bolyai-Lobachevskii-Gauss? Unfortunately, we didn't hear or talk about how Gauss got here. Or where are the German colleagues who would defend Gauss or explain Gauss's position regarding the discovery of non-Euclidean geometry.

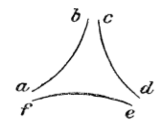
GAUSS-BOLYAI.

Göttingen 1832. III. 6.

Durch Deine beiden, mir durch Herrn *Zeyk* zugestellten Briefe hast Du, mein alter unvergesslicher Freund, mich sehr erfreuet. Ich zögerte nach Empfang des Ersten, Dir sogleich zu antworten, weil ich erst die Ankunft der versprochenen kleinen Schrift erwarten wollte, ausserdem auch durch mancherlei Lebensverhältnisse in eine höchst trübe Stimmung versetzt war, welche im Laufe der Zeit gemindert werden kann, aber insofern die Ursachen fortdauern, schwerlich vor meinem Ende ganz verschwinden wird.

Seit jener Zeit ist denn in meinen Lebensverhältnissen eine Hauptepoche eingetreten. Ich habe meine zweite Gattin, mit der ich 21 Jahre verbunden war, durch den Tod verloren. Den grössten Theil jener ganzen Zeit hatte sie gekränkelt; seit den letzten 9 Jahren aber hat sie, mit abwechselnden Erleichterungen, unbeschreiblich gelitten. Wie schwer ein solches Leiden drückt, und wie manche Nebenleiden im Gefolge davon erscheinen, brauche ich Dir nicht zu sagen, da Du Ähnliches erlebt hast. Wenn ich ihr nun Glück wünschen darf, von den Leiden endlich befreit zu sein: so fühle ich mich selbst dagegen nun so alleinstehend! —

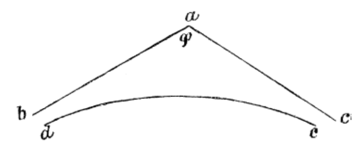
geschlagen: als ein Specimen füge ich einen rein geometrischen Beweis (in den Hauptzügen) von dem Lehrsatze bei, dass die Differenz der Summe der Winkel eines Dreiecks von 180° dem Flächeninhalte des Dreiecks proportional ist.



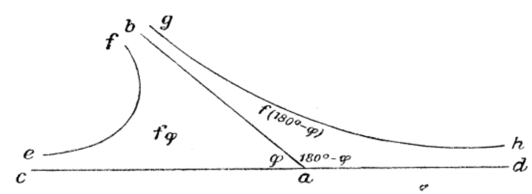
I. Der Complexus dreier Geraden *ab, cd, ef*, die so beschaffen sind dass *ab///dc, cd///fe, ef///ba*, bildet eine Figur, die ich *T* nenne. Es lässt sich beweisen, dass solche immer in einem Planum liege.

II. Derjenige Theil des Planums, welcher zwischen (*) den drei Geraden *ab, cd, ef*, liegt, hat eine bestimmte endliche Area: sie heisse *t*.

III. Indem zwei Geraden *ab, ac* sich in *a* unter dem Winkel φ schneiden, möge eine dritte Gerade *de* so beschaffen sein, dass *ab///ed, ac///de*: es liegt dann auch *de* mit *ab* u. *ac* in Einem Planum und die Area der Fläche zwischen diesen Geraden ist endlich, und nur von dem Winkel φ abhängig; offenbar bilden in Σ, *de* und *bac* nur Eine gerade Linie, wenn $\varphi = 180°$ ist, und folglich verschwindet der Werth jener Area mit $180° - \varphi$: man setze also allgemein die Area $= f(180° - \varphi)$, wo f ein Functionalzeichen bezeichnet.



IV. *Lehrsatz.* Es ist allgemein $f\varphi + f(180° - \varphi) = t$.

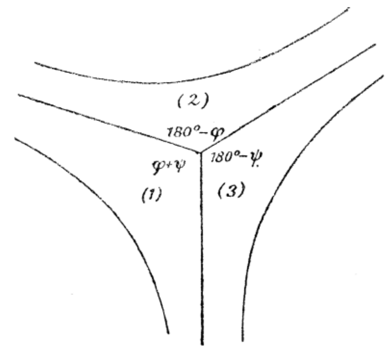


(*) Bei einer vollständigen Durchführung müssen solche Worte, wie «zwischen» auch erst auf klare Begriffe gebracht werden, was sehr gut angeht, was ich aber nirgends geleistet finde.

Let's look at the facts: Gauss's letter written in March 1832, the picture of which I am showing, and which he wrote to Farkas Bolyai, really testifies to such a great expertise and knowledge of the subject that we must say that only someone who had already thoroughly immersed himself could give such a professional answer in the subject!

Den Beweis gibt die Figur, wo $bac = \varphi$, $bad = 180° - \varphi$, $ac///fe$, $ef///ab$, $ab///hg$, $ad///gh$, und wo der Flächeninhalt roth eingeschrieben ist.

V. *Lehrsatz.* Es ist allgemein $f\varphi + f\psi + f(180° - \varphi - \psi) = t$. Der Beweis erhellet leicht aus der Figur, wo die drei Flächentheile (1), (2), (3) die Werthe haben

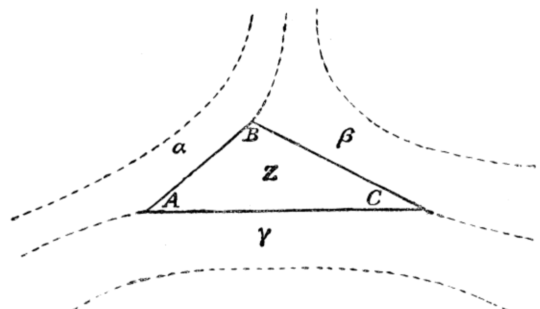


$$(1) = f(180° - \varphi - \psi)$$
$$(2) = f\varphi$$
$$(3) = f\psi$$

und ihre Summe $= t$ wird.

VI. Corollarium. Es ist also $f\varphi + f\psi = t - f\ (180° - \varphi - \psi) = f\ (\varphi + \psi)$: woraus leicht folgt dass $\frac{f\varphi}{\varphi} = Constans$, und zwar $= \frac{t}{180°}$ ist.

VII. *Lehrsatz.* Der Flächeninhalt eines Dreiecks, dessen Winkel A, B, C sind, ist $= \frac{180° - (A+B+C)}{180°} \times t$.



Den Beweis gibt die Figur. Es ist nämlich

$$\text{der Inhalt } \alpha = fA = \frac{A}{180°} \cdot t$$
$$\beta = fB = \frac{B}{180°} \cdot t$$
$$\gamma = fC = \frac{C}{180°} \cdot t$$
$$t = \alpha + \beta + \gamma + Z = \frac{A+B+C}{180°}\ t + Z.$$

And it is not only us Hungarians who say this, but one of the excellent constructive geometers of the 20th century, the Canadian COXETER, also says this. COXETER also believes that Hungarians should be proud of this letter and not interpret it with a negative sign.

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HISTORIA MATHEMATICA 4 (1977), 379-396

GAUSS AS A GEOMETER

BY H. S. M. COXETER,
UNIVERSITY OF TORONTO, TORONTO M5S 1A1

This paper was presented on 4 June 1977 at the Royal Society of Canada's Gauss Symposium at the Ontario Science Centre in Toronto.

SUMMARIES

In an attempt to reveal the breadth of Gauss's interest in geometry, this account is divided into six chapters. The first mentions the fundamental theorem of algebra, which can be proved only with the aid of geometric ideas, and in return, an application of algebra to geometry: the connection between the Fermat primes and the construction of regular polygons. Chapter 2 shows his essentially 'modern' approach to quaternions. Chapter 3 is a sample of his work in trigonometry. Chapter 4 deals with his approach to the geometry of numbers. Chapter 5 sketches his differential geometry of surfaces: his use of two parameters, the elements of distance and area, his theorema egregium, and the total curvature of a geodesic polygon. Finally, Chapter 6 shows that he continually returned to the subject of non-Euclidean geometry, which

6,50 x 9,50 in

1

18

Gauss defines parallel lines by letting a ray rotate clockwise about *A*, beginning with the position *AB* (Figure 5), so that the angle at *A* gradually increases. Among such rays, there is no last one that meets the ray *BN*, but there *is* a first one that *fails* to meet *BN*. This ray *AM* is said to be *parallel* to *BN*. In other words, *AM* is the Dedekind cut between the rays that meet *BN* and those that do not. He extends the notion from rays to lines by proving that *A* can be replaced by any other point on the line *AM*, and *B* by any other point on the line *BN*. He proves that this relation of parallelism is symmetric and transitive, and that parallel lines do not meet when extended backwards. The details are straightforward but tricky, making use of Pasch's ideas on order, long before Pasch was born [Coxeter 1969, 176-190, 265-268].

Two months later, Gauss wrote to Schumacher again, pointing out an error in the latter's 'proof' that the angle sum of a triangle is π. In non-Euclidean geometry, he said, there are no similar figures that are not congruent. It is possible for all three angles of a triangle to be zero, in which case one might draw it as in Figure 6.

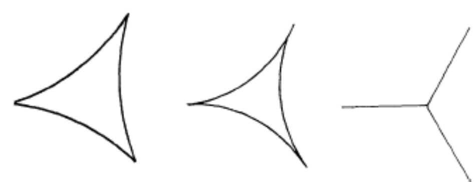


FIGURE 6

(The middle one of these three versions is almost the way Poincaré would have drawn it in his inversive model, fifty years later!)

Referring to Figure 7, Gauss remarked that, as *C* recedes from *A*, the difference

$$\sphericalangle DBC - \sphericalangle DAC$$

does not tend to zero (as it would in Euclidean geometry). In this remark he came close to Lobachevsky's proposed test for the nature of astronomical space, using parallax [Bonola 1906, 94].

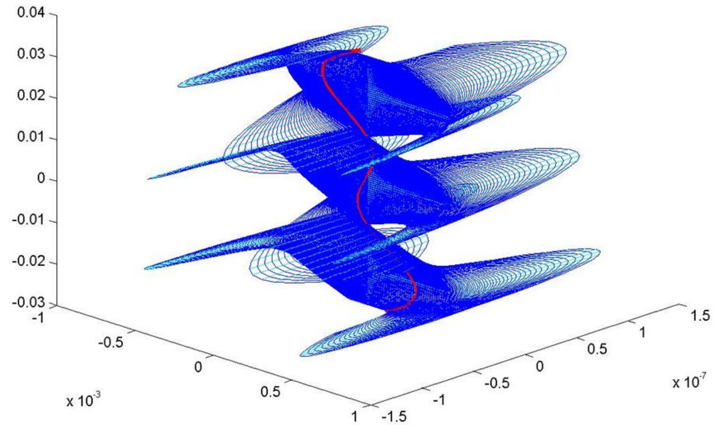
Another significant remark: in non-Euclidean geometry a

Indeed, Gauss writes that he is very happy that his good friend's son is the one who preceded him in this discovery! Furthermore, Gauss gave the names hypercycle and paracycle. And we can say that Gauss's comment is decisive, that although the area of the triangle on the hyperbolic plane can be derived very easily, its spatial analogy, the cubic content of the tetrahedron, is already hopelessly difficult. And indeed, the derivation of the cubic content of the tetrahedron was only given long after the death of Lobacsevskij and János Bolyai!

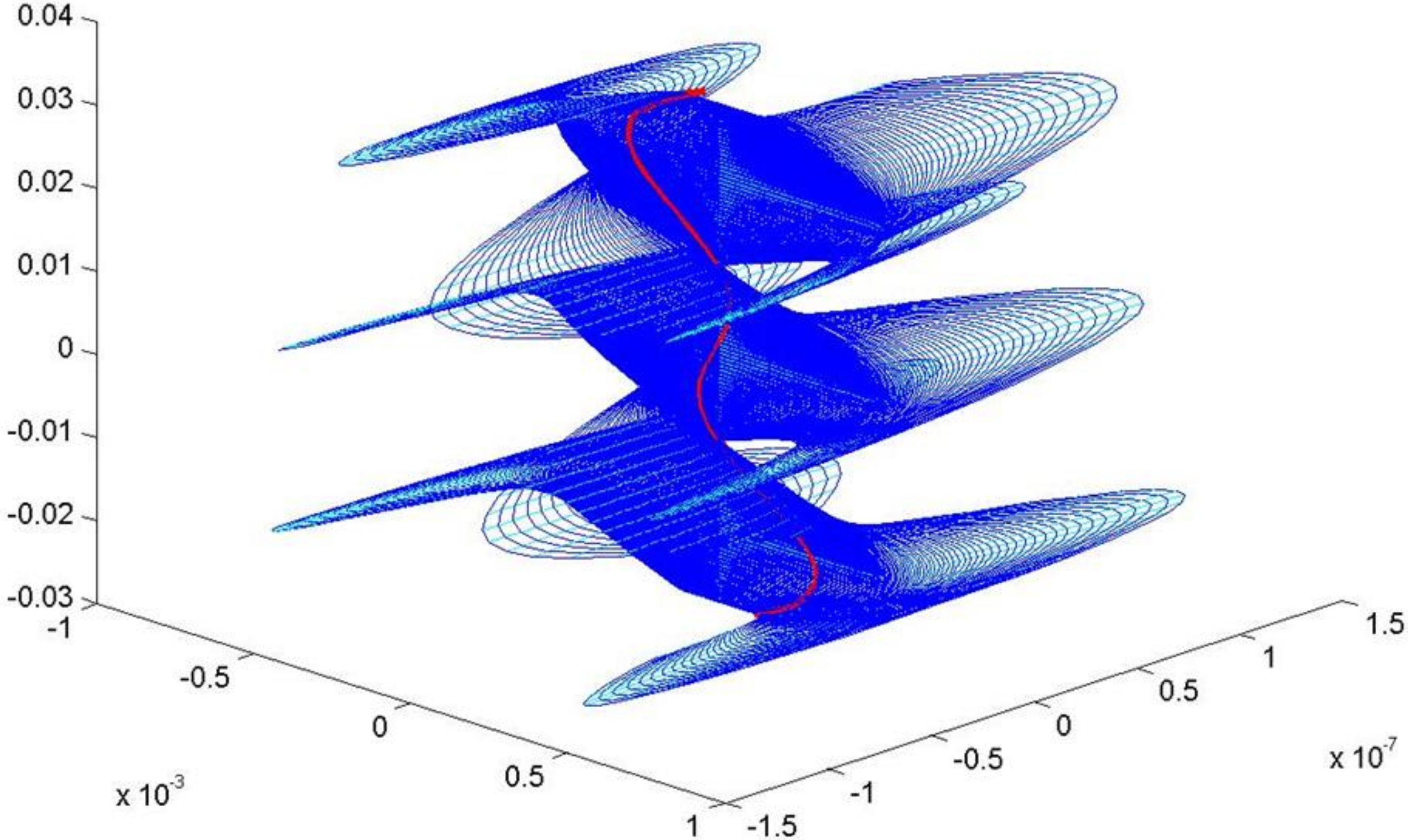
The volume of given tetrahedra in hyperbolic and spherical space is determined by their solid angles. The relationship is given by the Murakami–Yano formula. Since in Euclidean geometry, the angles of the tetrahedron are defined only to the extent of similarity, the formula cannot be applied in Euclidean space.

Murakami, Jun & Yano, Masakazu (2005), "On the volume of a hyperbolic and spherical tetrahedron", Communications in Analysis and Geometry 13 (2): 379–400, MR2154824, ISSN 1019-8385,

Finally, some interesting pictures that can only be illustrated with computer graphics: The hyperbolic plane can be embedded in the 6-dimensional Euclidean space, as Danilo Blanusa, a Croatian mathematician, showed in 1955.

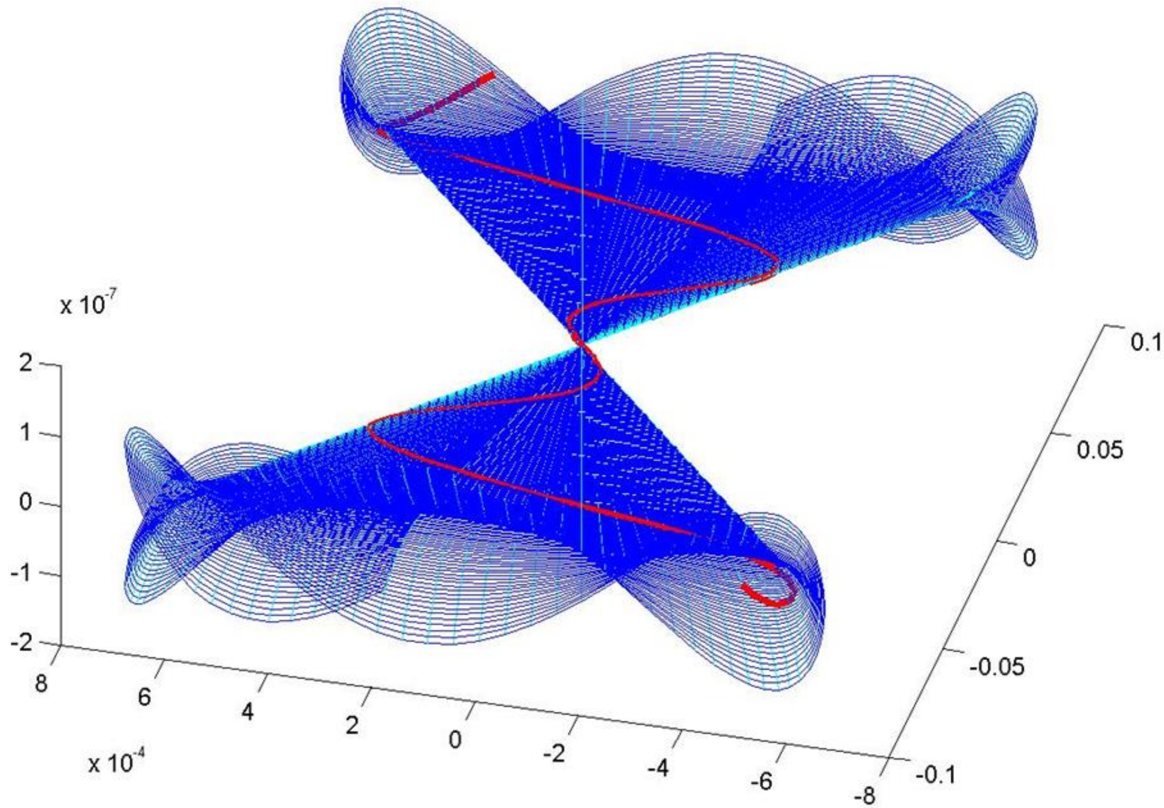


The Russian mathematician Efimov generalized Hibert's theorem by proving that not only a complete surface with negative constant curvature does not exist in 3-dimensional space, but also a complete surface in 3-dimensional Euclidean space whose curvature Sup K(u,v) < 0 does not exist. So the curvature supremum is strictly less than zero.

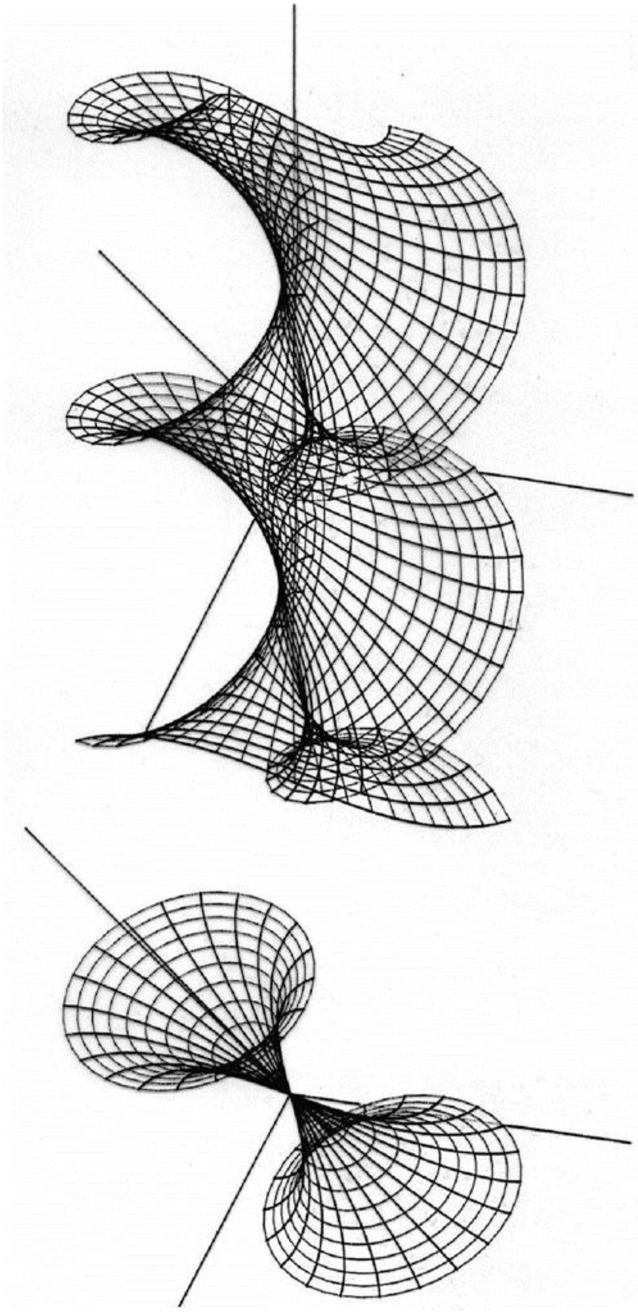


Danilo Blanusa's complete hyperbolic plane in 6-dimensional Euclidean space! Attention, this is a computer option to project from 6 dimensions into 3 dimensions. This is just an illustration of what computer graphics can do

The entire hyperbolic plane does not fit in 3-dimensional Euclidean space!



An immersion of Balnusa's former embedding into 5-dimensional Euclidean space (author's implementation)



The embedding of a region of the Bolyai plane into the 4-dimensional Euclidean space and its parallel projection into the 3-dimensional space. Implementation of OGR. My edit!

Thank you very much for your respectful attention!