BGL-2024
XII Bolyai–Gauss–Lobachevsky Conference
Budapest, 2nd May, 2024

Constructing slowly and rapidly rotating equilibrium configurations of relativistic stars

Dániel Barta
barta.daniel@wigner.hun-ren.hu

In collaboration with: Balázs Kacskovics, Mátyás Vasúth (†)

HUN REN | WIGNER

Related papers: arXiv:2212.04885 & 1908.02808

Supported by NKFIH under OTKA grant agreement No. K138277

Motivation

1) Additional angular velocity can counteract the extra gravitational force ⇨ Rotating compact stars can support a ***larger mass*** than their non-rotating counterparts.

- For slowly and uniformly rotating equilibrium solutions in a *Hartle–Thorne approximation* (quartic order in the angular velocity).
- For rapidly and uniformly rotating stars, we solve the *coupled system of non-linear elliptic PDEs* that are associated with the Einstein field equations (by implementing multi-domain spectral methods in the `LORENE/rotstar` codes).

To study the observable parameters of rotating relativistic compact stellar models based on the angular velocity and on the equations of state.

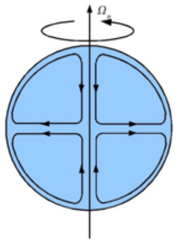
2)

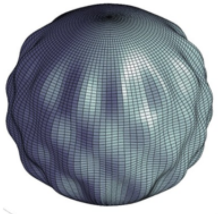
"Burst" emission

- Binary neutron star mergers (our safest bet for detection)
- Magnetar flares (likely too weak)
- Pulsar glitches (likely too weak)

Continuous emission

- Non-axisymmetric mass quadrupole ("mountains")
- Fluid part (oscillations)





Oscillation modes are *unstable* to gravitational wave emission
→ ***r-mode*** or ***f-mode*** oscillations

fundamental laws of conservation of energy and momentum

Stellar structure model in hydrostatic equilibrium

Energy–momentum tensor (perfect fluid):

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$$

The energy density and the pressure of the fluid are related by an **equation of state**:

$$p = p(\rho) \quad (T = 0)$$

Description of the state of matter

Metric tensor: $ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$
where $m(r) \equiv r(1 - e^{-\lambda})/2$ is the „gravitational mass" inside radius r

We are searching for three equations, which come from some combination of **equation of local conservation of energy and momentum ($\nabla_\mu T^{\mu\nu} = 0$)** and the **Einstein equations** ($G_{\mu\nu} = 8\pi T_{\mu\nu}$):

Gravitational mass: $\frac{dm}{dr} = 4\pi r^2 \rho$

Gravitational potential: $\frac{d\nu}{dr} = \frac{2m + 8\pi r^3 p}{r(r - 2m)}$

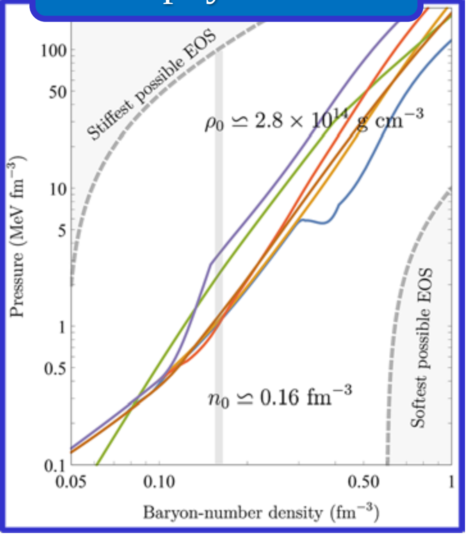
Hydrostatic equilibrium: $\frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r^2(1 - 2M/r)}$

(Tolman–Oppenheimer–Volkoff equation)

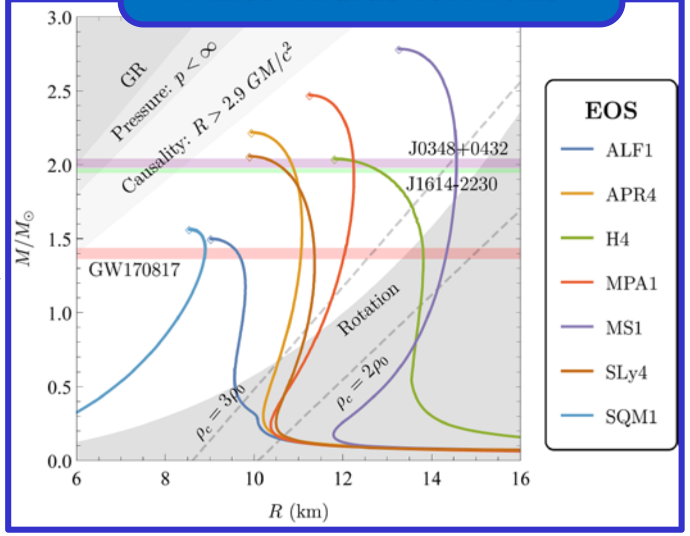
Relativistic corrections

Structure

Microphysics: EOS



Macroscopic observables: Mass–radius relations



At the stellar center ($r = 0$):

- $M(0) = 0$: the mass function vanish
- $\rho_0 \equiv \rho(0)$: central density is freely specified

At the stellar surface ($r = R$):

- $M \equiv m(R)$: total mass of the star
- $p(R) = 0$: the isotropic pressure vanishes
- $e^{\nu(R)} = 1 - 2M/R$: normalizing the time coordinate at spatial infinity

Boundary conditions

Hartle–Thorne slow-rotation approach

Exact solution of Einstein's equations describing spacetime in the vicinity of a ***perfect fluid***, ***stationary*** and ***axially symmetric*** and ***slowly rotating star***:

Hartle (1967), Hartle–Thorne (1968), Chandrasekhar–Miller (1974), Miller (1977):

- Slow-rotation approximation: $\Omega^2 \ll GM/R^3 = \Omega^2_{\text{Kepler}}$ (or mass-to-radius ratio $GM/c^2/R \gtrsim 0.1$)
- Terms up to 2nd order in Ω are taken into account

2nd-order Legendre polynomial: $P_2(\cos\theta) = (3\cos^2\theta - 1)/2$

$$ds^2 = e^{2\nu_0}[1 + 2h_0(r) + 2h_2(r)P_2(\cos\theta)]dt^2 + e^{2\lambda_0}\left\{1 + \frac{e^{2\lambda_0}}{r}[2m_0(r) + 2m_2(r)P_2(\cos\theta)]\right\}dr^2 + r^2[1 + 2k_2(r)P_2(\cos\theta)]\{d\theta^2 + [d\phi - \omega(r)dt]^2\sin^2\theta\}$$

- $\omega(r)$ – 1st order in Ω
- $h_0(r),\ h_2(r),\ m_0(r),\ m_2(r),\ k_2(r)$ – 2nd order in Ω, functions of r

Parameters that fully describing the star within HT approx.

Within the slow rotation approximation only quantities up to 2nd order in Ω are taken into account:

- J – specific angular momentum
- M – total gravitational mass
- Q – dimensionless quadrupole moment

1. Computation of angular momentum

From $(t\varphi)$ component of Einstein equation

$$\frac{1}{r^3}\frac{d}{dr}\left(r^4 j(r)\frac{d\tilde{\omega}}{dr}\right)+4\frac{dj}{dr}\tilde{\omega}=0$$

$$\tilde{\omega}(r)=\Omega-\omega(r) \qquad j=e^{-(\lambda_0+\nu_0)}$$

- Equation is solved with proper boundary condition
- We want to calculate models for a given Ω – rescaling

$$J=\frac{1}{6}R^4\left(\frac{d\tilde{\omega}}{dr}\right)_{r=R}, \qquad I=\frac{J}{\Omega}$$

2. Computation of mass

Calculation of the spherical perturbation ($l=0$) quantities:

$m_0(r)$: $$\frac{dm_0}{dr}=4\pi r^2(\rho+p)\frac{d\rho}{dp}\delta p_0+\frac{1}{12}r^4 j^2\left(\frac{d\tilde{\omega}}{dr}\right)^2-\frac{1}{3}r^3\tilde{\omega}^2\frac{dj^2}{dr}$$

$p_0(r)$: $$\frac{dp_0}{dr}=-\frac{m_0(1+8\pi r^2 p)}{(r-2m)^2}-\frac{4\pi(\rho+p)r^2}{r-2m}p_0+\frac{1}{12}\frac{r^4 j^2}{r-2m}\left(\frac{d\tilde{\omega}}{dr}\right)^2+\frac{1}{3}\frac{d}{dr}\left(\frac{r^3 j^2\tilde{\omega}^2}{r-2m}\right)$$

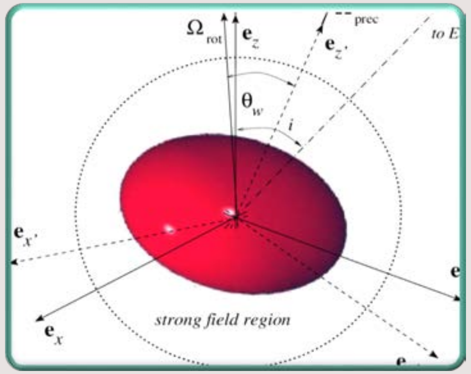
- Total gravitational mass of the rotating star:

$$M(R)=M_0(R)+m_0(R)+J/R^3$$

3. Computation of quadrupole moment: Calculation of the deviation from spherical symmetry

$$\frac{dv^2}{dr}=-\frac{2d\nu_0}{dr}h_2+\left(\frac{1}{r}+\frac{d\nu_0}{dr}\right)\left[\frac{1}{6}r^4 j^2\left(\frac{d\tilde{\omega}}{dr}\right)^2-\frac{1}{3}r^3\tilde{\omega}^2\frac{dj^2}{dr}\right]$$

$$\frac{dh_2}{dr}=-\frac{2v^2}{r\big(r-2m(r)\big)\frac{d\nu_0}{dr}}+\left\{-2\frac{d\nu_0}{dr}+\frac{r}{r\big(r-2m(r)\big)\frac{d\nu_0}{dr}}\left[8\pi(\rho+p)-\frac{4m(r)}{r}\right]\right\}h_2$$
$$+\frac{1}{6}\left[r\frac{d\nu_0}{dr}-\frac{1}{2\big(r-2m(r)\big)\frac{d\nu_0}{dr}}\right]r^3 j^2\left(\frac{d\tilde{\omega}}{dr}\right)^2-\frac{1}{3}\left[r\frac{d\nu_0}{dr}+\frac{1}{2\big(r-2m(r)\big)\frac{d\nu_0}{dr}}\right]r^2\tilde{\omega}\frac{dj^2}{dr}$$



$Q=\frac{8}{5}KM^3+\frac{J^2}{M}$ where K comes from matching of internal and external solutions

Stationary and axisymmetric approach

Symmetries

We suppose that there exists two Killing vector fields:

- $\vec{\xi}$ (timelike) *to account for **stationarity***;
- $\vec{\chi}$ (spacelike) with closed orbits for **axisymmetry**

Quasi-isotropic coordinates

The coordinates (t,r,θ,φ) with an only (r,θ)-dependent line element are called quasi-isotropic coordinates.

Under such conditions, it is possible to choose adapted coordinates, such that the *metric depends only on two coordinates (r,θ)* and takes the following form:

$$ds^2 = -N^2 dt^2 + A^2(dr^2 + r^2 d\theta^2) + B^2 r^2 \sin^2\theta (d\varphi - \omega dt)^2$$

$B = B(r,\theta)$ is defined by $B^2 = \frac{g_{\varphi\varphi}}{r^2 \sin^2\theta}$

$\omega = \omega(r,\theta)$ is defined as the normalized scalar product of the two Killing vectors:

$$\omega \equiv -\frac{\vec{\xi}\cdot\vec{\chi}}{\vec{\chi}\cdot\vec{\chi}} \Longrightarrow \begin{matrix} g_{t\varphi} = \vec{\xi}\cdot\vec{\chi} \\ g_{\varphi\varphi} = \vec{\chi}\cdot\vec{\chi} \end{matrix} \Longrightarrow g_{t\varphi} = -\omega g_{\varphi\varphi}$$

The minus sign ensures that for a rotating star, ω≥0

$A = A(r,\theta)$ is defined by $g_{ab}dx^a dx^b = A^2(dr^2 + r^2 d\theta^2)$

All metrics are ***conformally related*** in 2 dimensions. They differ from each other only by a scalar factor A^2.

Field equations in QI coordinates

In this gauge, the Einstein's field equations for ***rigidly rotating stars*** at the frequency Ω turn into a system of *four coupled non-linear elliptic partial differential equations*:

NON-LIN. ELLIPTIC PDEs

$$\Delta_3 \nu = 4\pi A^2 (E + 3p + (E+p)U^2) + \frac{B^2 r^2 \sin^2\theta}{2N^2} (\partial\omega)^2 - \partial\nu\,\partial(\nu + \beta)$$

$$\tilde{\Delta}_3 (\omega r \sin\theta) = -16\pi \frac{NA^2}{B} (E+p)U - r\sin\theta\,\partial\omega\,\partial(3\beta - \nu)$$

$$\Delta_2 [(NB-1)\, r\sin\theta] = 16\pi N A^2 B p r \sin\theta$$

$$\Delta_2 (\nu + \alpha) = 8\pi A^2 [p + (E+p)U^2] + \frac{3B^2 r^2 \sin^2\theta}{4N^2} (\partial\omega)^2 - (\partial\nu)^2$$

DIFFERENTIAL OPERATORS

$$\Delta_2 := \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$\Delta_3 := \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2 \tan\theta}\frac{\partial}{\partial\theta}$$

$$\tilde{\Delta}_3 := \Delta_3 - \frac{1}{r^2 \sin^2\theta}$$

$$\partial a \partial b := \frac{\partial a}{\partial r}\frac{\partial b}{\partial r} + \frac{1}{r^2}\frac{\partial a}{\partial\theta}\frac{\partial b}{\partial\theta}\,.$$

Laplacian in a 2-dimensional flat space (Δ_2)

Laplacian in a 3-dimensional flat space (Δ_3)

with the following notations: $\nu := \ln N,\ \alpha := \ln A,\ \beta := \ln B$

- fluid 3-velocity in the φ-direction: $U = Br\sin\theta(\Omega - \omega)/N$
- total energy density: $E = \Gamma(\varepsilon + p) - p$

Both measured by a locally non-rotating observer

$\Gamma = \sqrt{1 - U^2}$ – Lorentz factor

Using log-enthalpy

A *perfect fluid* at *zero temperature* is a ***good approximation*** for a neutron star (except immediately after its birth)

Stress–energy tensor (perfect fluid):

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + p g^{\mu\nu}$$

where u^μ is the fluid 4-velocity, p its pressure and ε its total energy density.

EOS (T=0):

$\varepsilon = \varepsilon(n_\mathrm{b})$

$p = p(n_\mathrm{b})$

⇨

Conservation laws

Energy–momentum conservation: $\nabla_\mu T^{\alpha\mu} = 0$

Baryon-number conservation: $\nabla_\mu (n_\mathrm{b} u^\mu) = 0$

- The only non-trivial hydrostationary equation is the ***relativistic Euler's equation of motion*** (which can be obtained from the spatial sector of the local energy–momentum conservation equation):

$$(\varepsilon + p)u^\mu \nabla_\mu u_\alpha + \left(\delta^\mu_\alpha + u^\mu u_\alpha\right)\nabla_\mu p = 0$$

- In the *stationary*, *axisymmetric* and *circular* case, Euler's equation ***turns into a simple first integral***:

$$\boxed{H + \nu - \ln\Gamma = \mathrm{const.}\ \text{(along a fluid line)}}$$

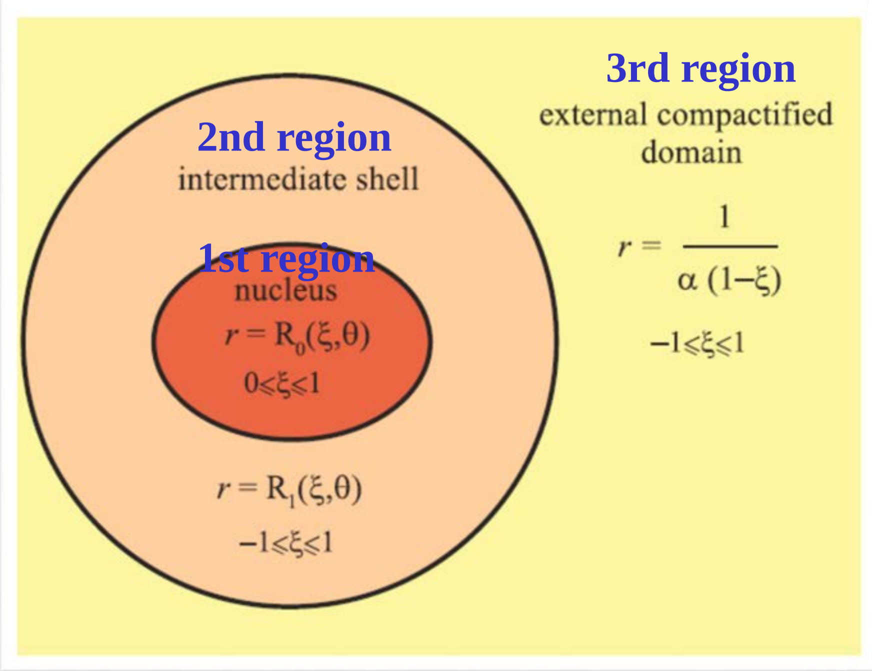
with the log-enthalpy

$$H = \ln\left(\frac{\varepsilon + p}{n_\mathrm{b} c^2}\right)$$

As before, notations for the metric function and the Lorentz factor: $\Gamma = \sqrt{1 - U^2}, \nu = \ln N$

LORENE (Langage Objet pour la REIativité NumériquE) is a set of C++ classes to solve various problems arising in numerical relativity, and more generally in computational astrophysics.

The computational domain of LORENE/rotstar is composed of three regions



1. The first region, the so-called ***nucleus***, is a spheroidal domain, for which the surface is adapted to the stellar surface.

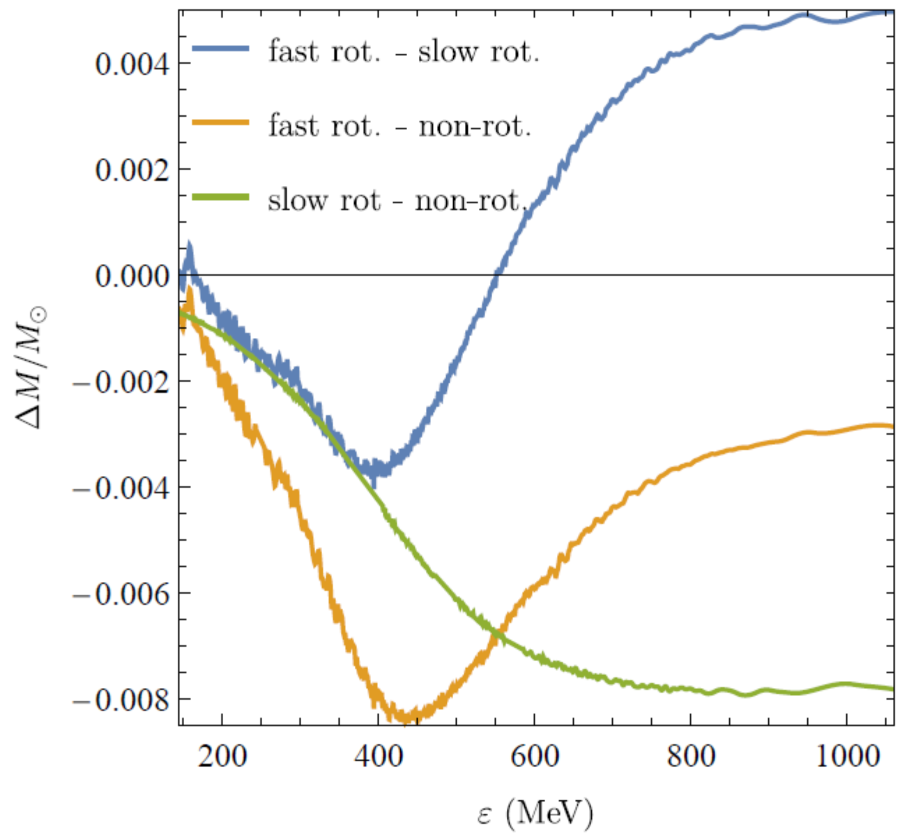
2. The second region is a ***shell region*** surrounding the nucleus. The inner boundary of this shell is the same as the outer boundary of the nucleus, while the outer boundary of the shell is a sphere with twice the radius of the nucleus at the equator.

3. The third region is a ***compactified external domain*** that extends from the outer boundary of the shell to spatial infinity. The compactified external domain allows us to impose exact boundary conditions at spatial infinity.

Solving the elliptic equations

- The ***elliptic equations*** are solved in each computational domain, and matching conditions are imposed so that values of the metric functions and their derivatives agree on both sides of each domain.
- In LORENE, functions of r and θ are expanded in ***Chebyshev polynomials*** and ***trigonometric functions***, respectively, and the latter are re-expanded in ***Legendre polynomials*** when it is advantageous.

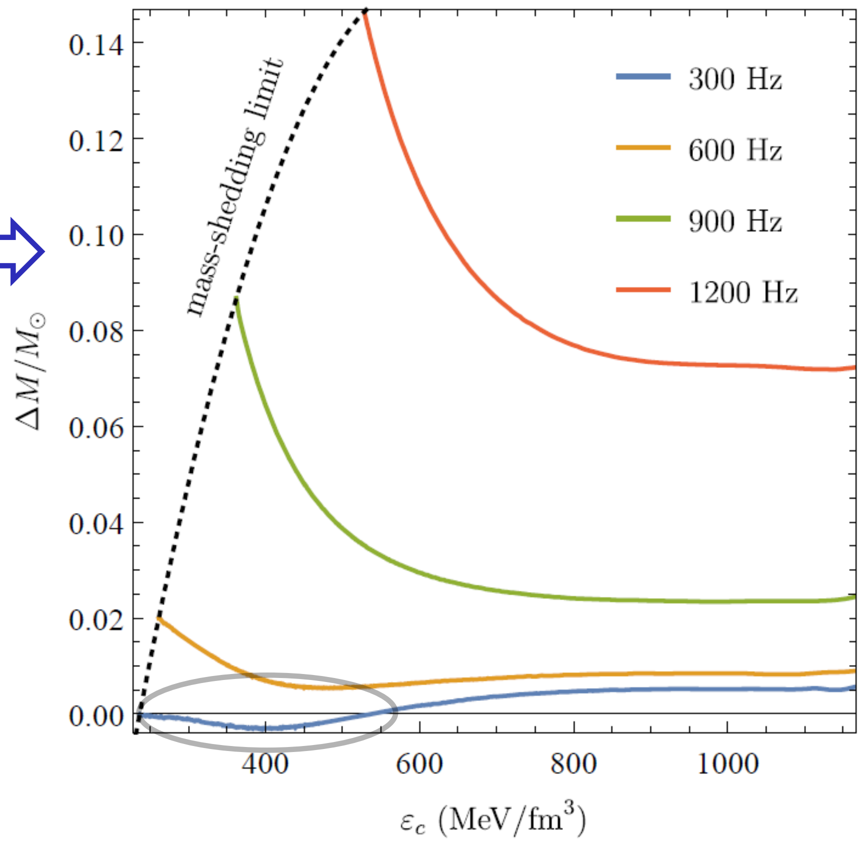
Static configurations computed by three different methods

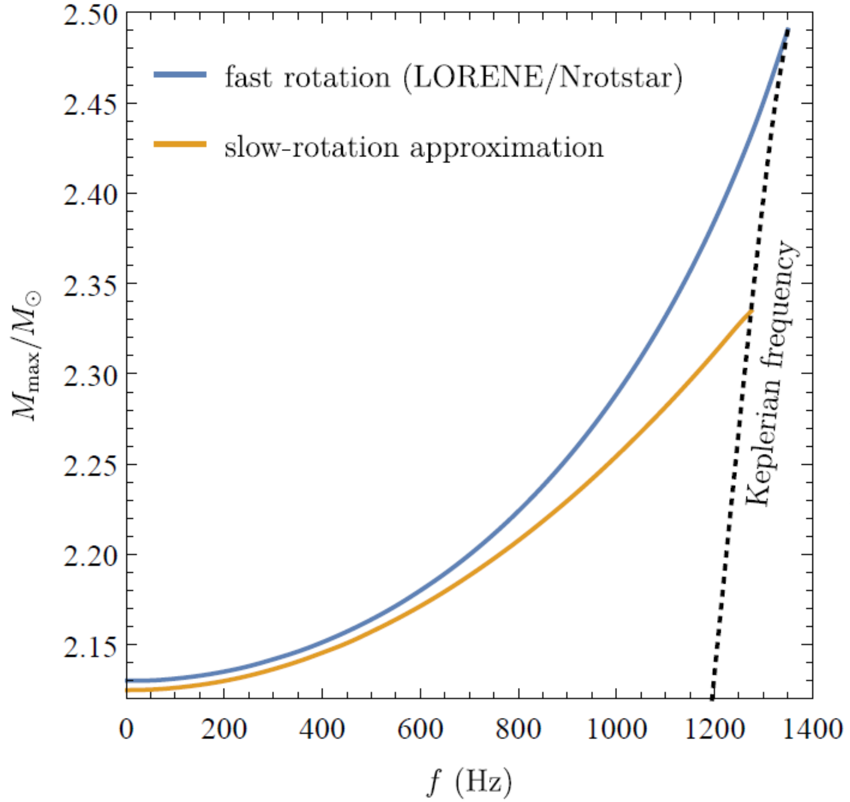


- At low energy densities both overestimate the mass compared to the one determined by LORENE.
- At higher energy densities the difference slightly decreases, and note that the slow-rotating approach starts to underestimate LORENE, which remains a characteristic feature of slow-rotating approach.

Departure from the slow-rotation approximation

- The mass-shedding limit imposes a lower limit on the ε_c at each frequency
- On low ε_c increasing departure from the slow-rotation approximation, as the frequency reaches the Keplerian limit
- At 300 Hz, the overestimation of the static case at low-energy density is still visible.





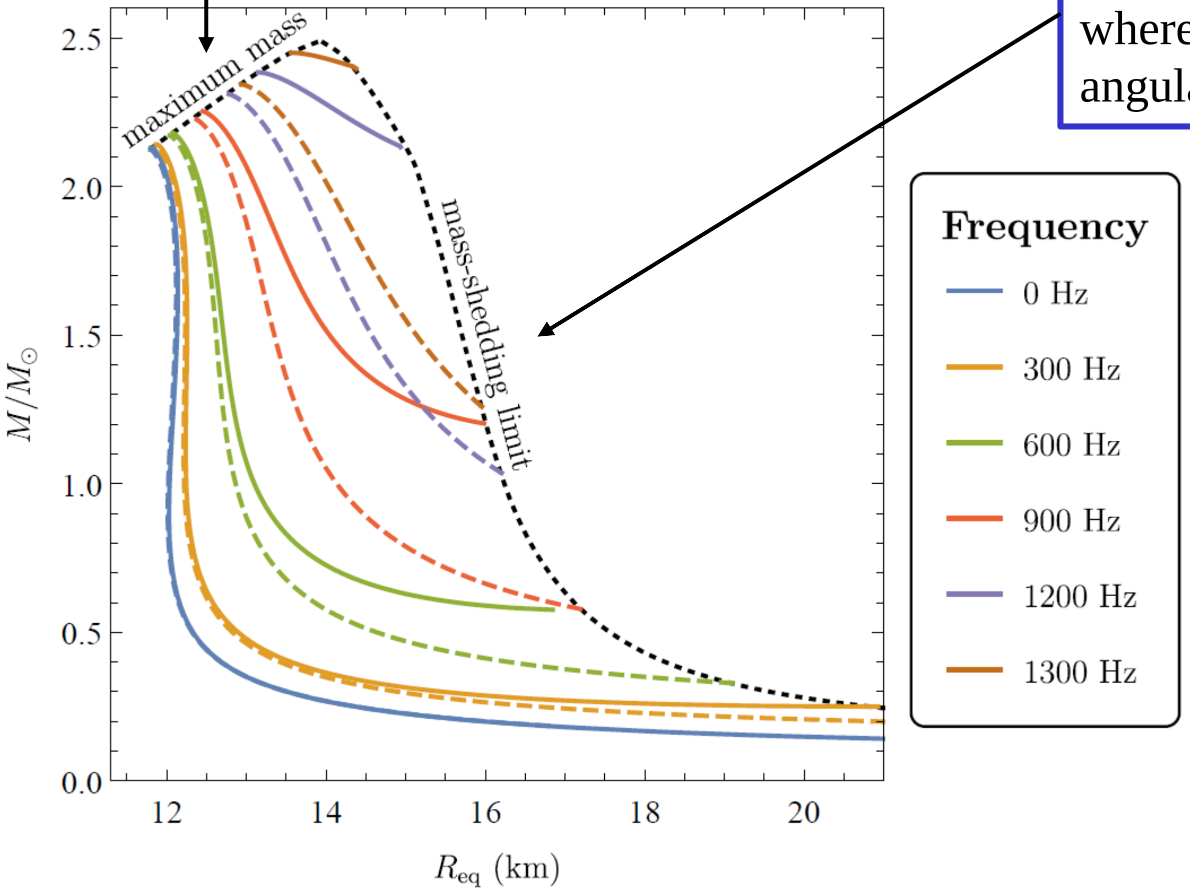
- As approaching f_K, the difference in the computed M_{max} grows at an increasing rate
- At the mass-shedding limit, the discrepancy between the two methods is 6.67%, and maximum masses are $2.34M_\odot$ and $2.49M_\odot$, respectively.

B. Kacskovics, D. Barta and M. Vasúth. *Astron. Nachr.*, **334**:220121 (2023)

Limits on the stability of rotating relativistic stars

Secular axisymmetric instability:

$\left(\frac{\partial M(\rho_c, J)}{\partial \rho_c}\right)_J = 0$: Turning-point method to locate the points where ***secular instability sets in*** for uniformly rotating relativistic stars.



Mass-shedding instability:

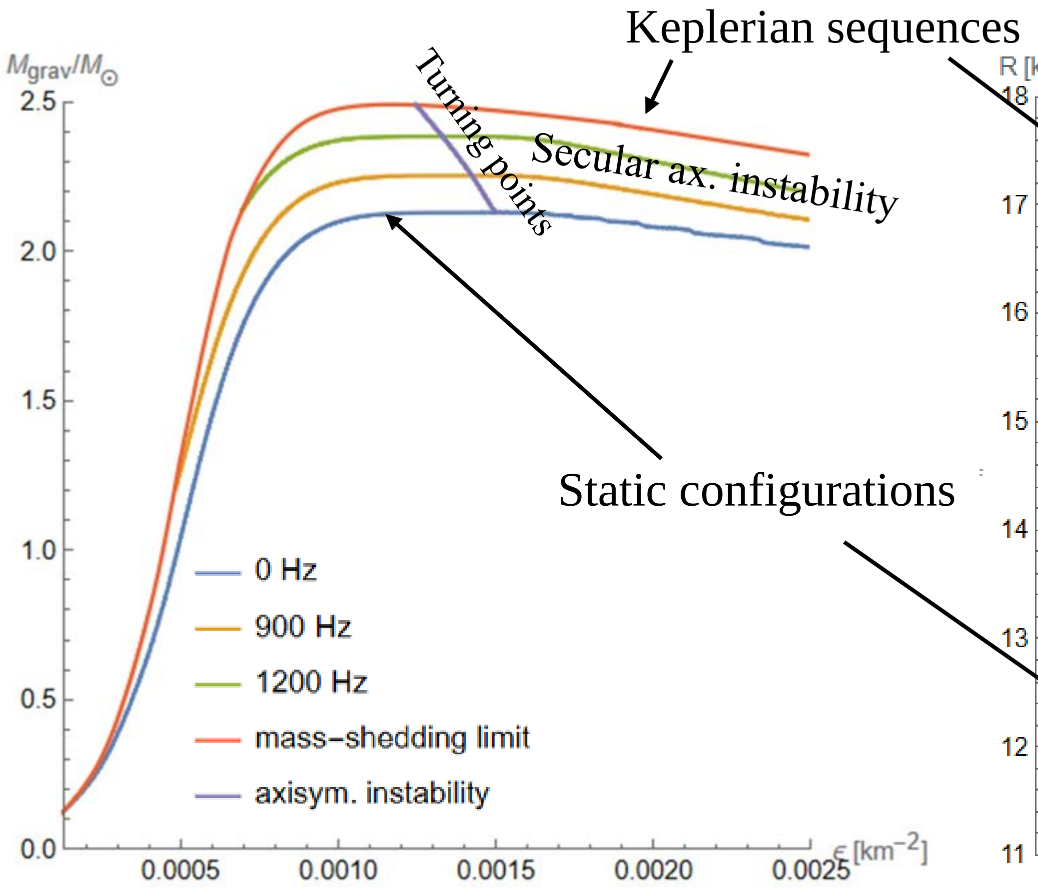
For the Hartle–Thorne external solution, the Keplerian (or mass-shedding) angular velocity can be written as:

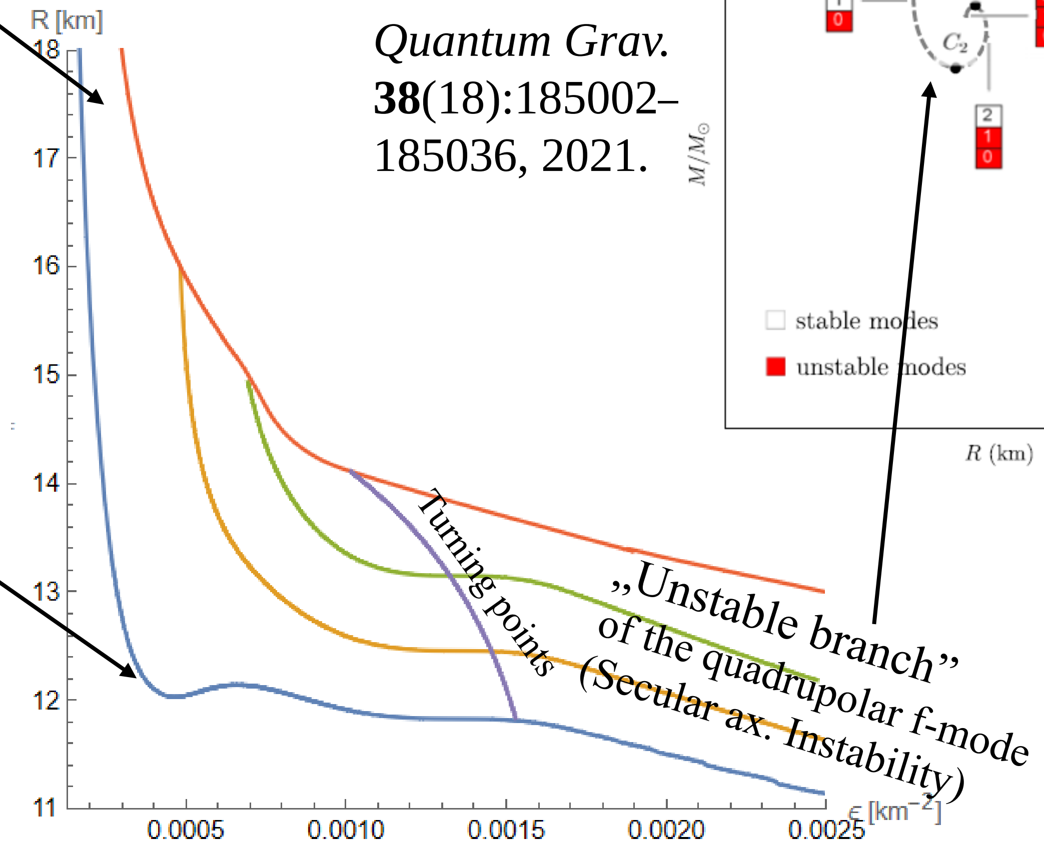
$$\Omega_K = \sqrt{\frac{GM}{R_{eq}{}^3}}\left[1 - jF_1(R_{eq}) + j^2F_2(R_{eq}) + qF_3(R_{eq})\right]$$

where $j = J/M^2$ and $q = Q/M^3$ are the dimensionless angular momentum and quadrupole moment.

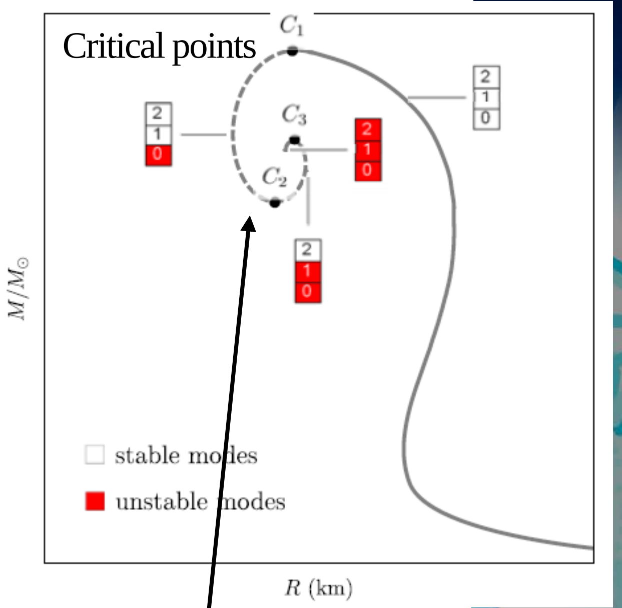
The ***solid lines*** represent sequences computed by LORENE, and ***dashed lines*** represent those of our slow-rotating HT model on different frequencies.

Boundary limits on observables: Gravitational mass & equatorial radius





D. Barta. *Class Quantum Grav.* **38**(18):185002–185036, 2021.



- For rotating stars, the turning point is a *sufficient* but *not a necessary condition* for ***instability***: The onset of instability is at a configuration with slightly lower ε_c (for fixed angular momentum) than that of the star with M_{max}. [Friedman & Stergioulas, 2013]

Current and future research

Inclusion of new EOS tables into **CompOSE**

Add new representative EOS tables into **CompOSE** → **LORENE/rotstar** loads tabulated EOS models in CompOSE format.

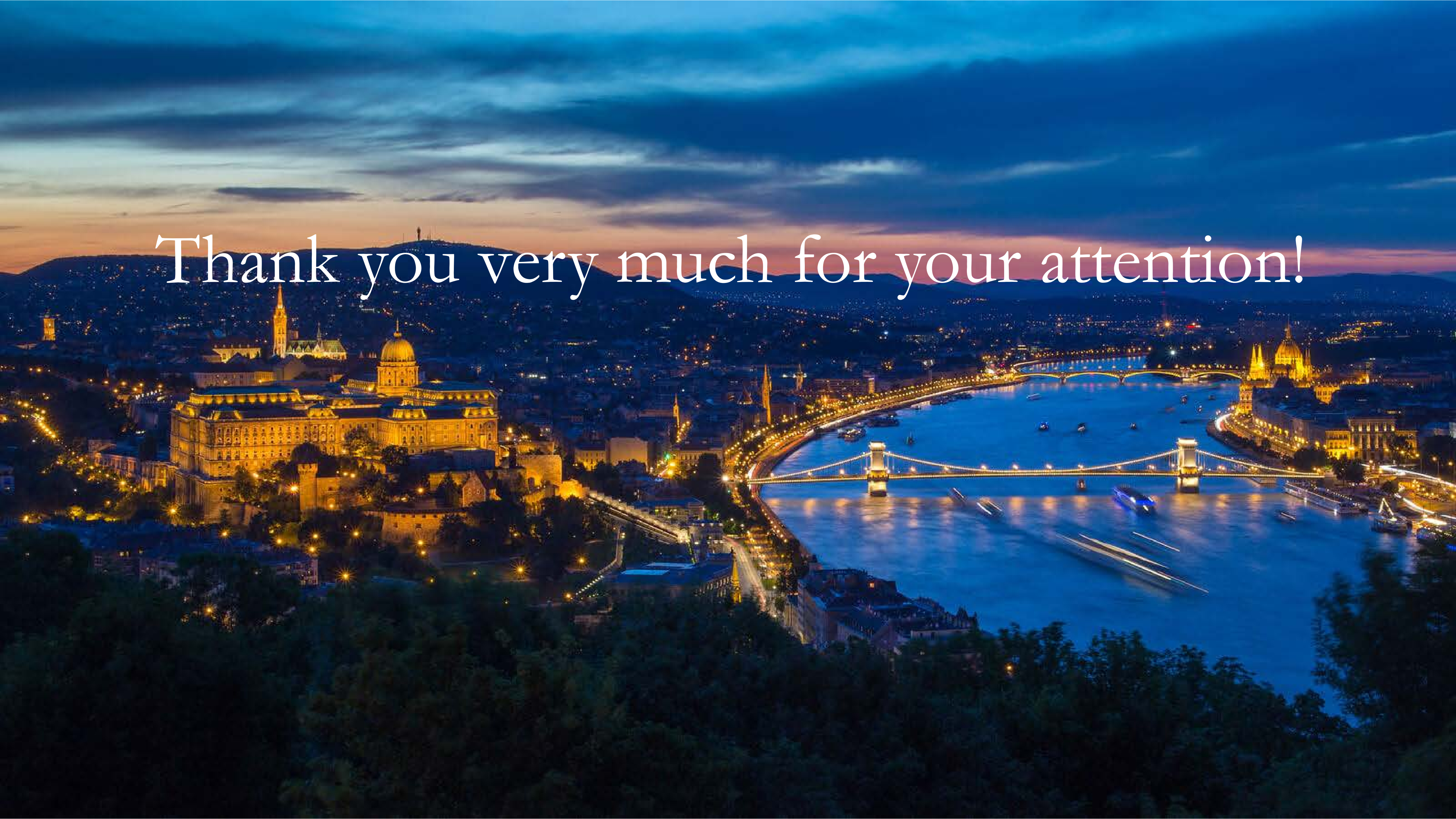
- CompOSE: online repository of EOS for use in nuclear physics and astrophysics

⇨ Exploration of the region of stable configurations for compact stars with various nucleonic and hybrid EOS in their cores.

Study of GW-radiating oscillation modes

The background quantities for fast-rotating stationary configurations will be computed by **LORENE/rotstar**. We assume small deviations for the fluid variables and study their linearized perturbations.

⇨ Neutron star oscillations as sources of gravitational waves: *f*- and *r*-mode oscillations


Thank you very much for your attention!

New equation of state (SFHo)

Axial-vector meson-extended quark–meson model describes the quark matter in the NS core.

A *perfect fluid* at *zero temperature* is a ***good approximation*** for a neutron star (except immediately after its birth)

Stress–energy tensor (perfect fluid):

$$T^{\mu\nu} = (\varepsilon + p)u^{\mu}u^{\nu} + p g^{\mu\nu}$$

where u^{μ} is the fluid 4-velocity, p its pressure and ε its total energy density.

EOS (T=0):

$\varepsilon = \varepsilon(n_{\mathrm{b}})$

$p = p(n_{\mathrm{b}})$

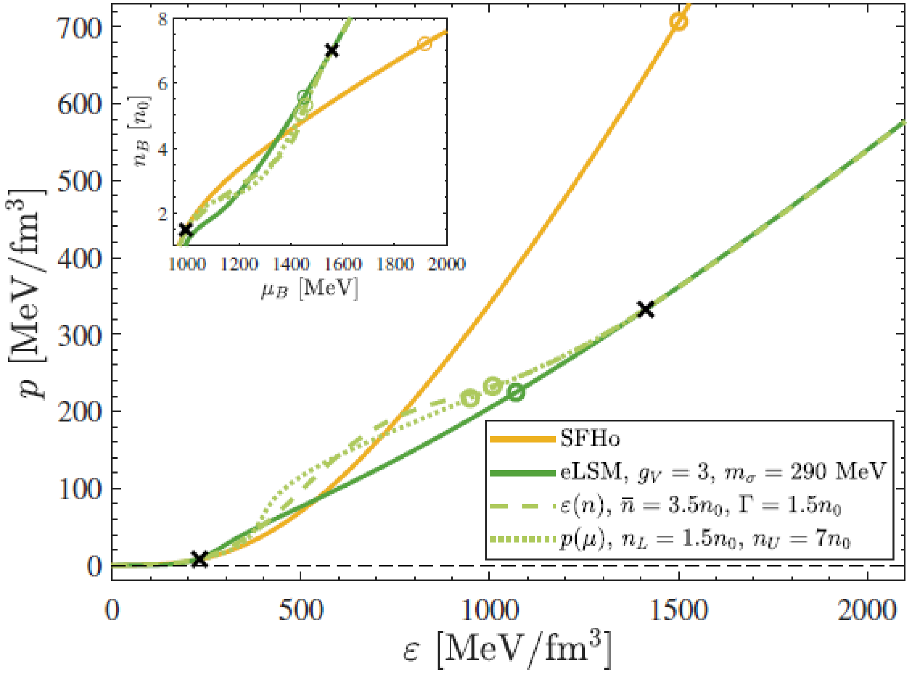
Conservation laws

Energy–momentum conservation: $\nabla_{\mu} T^{\alpha\mu} = 0$

Baryon-number conservation: $\nabla_{\mu}(n_{\mathrm{b}} u^{\mu}) = 0$

Property	SFHo	DD2
Saturation density, n_0 [fm^{-3}]	0.16	0.15
Binding energy per baryon, E_0 [MeV]	-16.17	-16.02
Compressibility, K_0 [MeV]	245.2	242.7
Symmetry energy, S_0 [MeV]	31.2	32.73
Slope of symmetry energy, L [MeV]	45.7	57.94
Maximum mass neutron star [$M_{\odot}$]	2.06	2.42
Radius of $M = 1.4\,M_{\odot}$ neutron star [km]	11.97	13.26

Table. Nuclear properties of symmetric nuclear matter described by the SFHo and DD2 RMF models as well as some properties of neutron stars described by these models.



P. Kovács, J. Takátsy, J. Schaffner-Bielich, and Gy. Wolf. *Phys. Rev.* D **105** (2022), 103014, arXiv:2111.06127