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QCD Inspired Dynamical SUSY based on SU(6/22), Split Octonion Algebra and new Mass Formulae

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A sketchy background of quark dynamics, necessitating "dynamical SUSY" is detailed, followed by an explanation of the hadronic color algebra based on split-octonions. We then move on to the evolution of new hadronic mass formulae with emphasis on group theoretical descriptions and SUSY suggested by QCD and based on quark-antidiquark symmetry.

Quarks were originally introduced as exotic entities, far removed from the observed particles in 1964. It turned out that only triplet combination of quarks (qqq) and antiquarks $(\bar{q}\bar{q}\bar{q})$ and also paired combinations $(q\bar{q})$ are realized in nature, resulting in a very simple scheme for classifying the observed baryons, antibaryons, and mesons. This scheme was based on the simple future of unitary groups: any representation of group $SU(N)$ can be obtained by the tensor multiplication of fundamental representations. Because the restrictions on the existence in nature of quarks themselves and other combinations of them such as (qq) were purely formal (and also because of difficulties related to violation of the Pauli theorem on the connection of spin and statistics), in the beginning, quarks served only as a device for the simple and elegant construction of all the hadrons with spin 0 and $\frac{1}{2}$, and also resonant states with spin 1 and $\frac{3}{2}$.
Quark model provided a natural explanation for the existence of multiplets. In fact, states composed of quark and anti-

quark (not necessarily the same type) can be constructed in nine different way, giving a meson singlet and octet. Similarly, the 27 combinations of three quarks form a singlet, two octets, and a decuplet of baryons.
This splitting of the various combinations into multiplets has a simple justification in the theory of group representations. Specifically, products of three-dimensional representations of $SU(3)$ can be decomposed into direct sums of representations of this group in the following manner: $3 \times \bar{3} = 1 + 8$; $3 \times 3 \times 3 = 1 + 8 + 8 + 10$.
All the hadrons known in 60's were successfully described by the three-quark model. In the early 70's to account for the suppression of certain weak decays, it was suggested that there also exists a fourth quark, carrying charm. Right after this came the discovery of J/ψ meson which is interpreted as a $c\bar{c}$ bound state. Later charmed D and F mesons were found. At present, we have six types of quarks, each quark having its own quantum number distinguishing it from others, which is now referred to as the quark flavor.

Quark model in the form of the above discussion encountered a fundamental difficulty relating to the existence of symmetric quark states. If quarks are assigned spin $\frac{1}{2}$ (required to obtain observed hadron spins), we come into a contradiction of the Pauli principle. Since this model represents certain particles as combinations of three identical quarks in the same state; for example, Δ^+ resonance with three $(u \uparrow u \uparrow u \uparrow)$ quarks with parallel spins (spin up), and the Ω^- hyperon with three strange quarks with parallel spins $(s \downarrow s \downarrow s \downarrow)$ (spin down). The solution was found by assigning that quarks carried some new quantum number, later called "color". The color of each quark can take three values. In this new picture the quarks forming hadrons, for example, Δ^+ and Ω^-, differ in color, so there is no conflict with the Pauli principle. Finally, the color symmetry group was postulated as $SU(3)^c$.

To make the story short, all these developments led to the existence of supersymmetry, leading to other major developments in physics. Particularly important are:

1. Kinematic (space-time) supersymmetries:
An example of this is Wess-Zumino invariance. Unfortunately, there is no experimental evidence yet for this type of supersymmetry.
2. Dynamic (internal) spersymmetries:
Which Gürsey and SC developed using supergroup $SU(6/21)$ to be described below. Originally Miyazawa used $U(6/21)$ and invented supersymmetry algebra long before other supersymmetric constructions were built by others. Starting from the generalization of $SU(3) \times SU(3)$ current algebra, he embedded it in a superalgebra that has $SU(6)$ as its bosonic subgroup and introduced anticommuting parameters for the supergroup. Later, he constructed the superalgebras we call $SU(m/n)$, giving the correct commutation and anticommutation relations among its even and odd generators, and discovered the generalization of the Jacobi identity.
There is a good phenomenological evidence that in a rotationally excited baryon a quark-diquark $(q-D)$ structure is fa-

vored over a three quark (qqq) structure. Regge trajectories for mesons and baryons are closely parallel; both have a slope of about 0.9 GeV^{-2}. At large spin two of the quarks form a diquark ($D = qq$), a bilocal object at one end of a bag, the remaining quark being at the other. For the light quarks Gürsey and I had shown that the underlying quark-diquark symmetry leads to supersymmetric $SU(6/21)$ symmetry between mesons and baryons, going beyond Miyazawa symmetry, in a modern context. This new scheme uses split octonion algebra that produces the algebraic description of color degrees of freedom, supresses color-symmetric and space-symmetric quark configurations, and leads to existence of exotic meson (diquark-antidiquark) states that are experimentally confirmed. Our results are readily expandable into explanations of tetraquark and multiquark states.

Further background:
Leading mesonic trajectories associated with the lowest spin $0, 1$, and 2 are parallel. Also leading baryonic trajectories with $s = \frac{1}{2}, \frac{3}{2}$ are similarly mutually parallel. These hint at a phenomenological symmetry between mesons of different spin and baryons of a different spin. For hadrons made up of three light flavor quarks u, d, and s, this symmetry is described by the group $SU(6) \times O(3)$. $SU(6)$ classifies the lowest elements of the trajectories into multiplets while $O(3)$ accounts for the rotational excitations on the leading trajectories. Mesonic and baryonic trajectories are nearly parallel to one another with a universal slope of $\alpha' = 0.9\ Gev^{-2}$. This is the hallmark of new symmetry, indeed a supersymmetry, between bosonic mesons and fermionic baryons. Here a manifest supersymmetric observable is the Regge slope, a universal constant of hadronic physics, apparently independent of flavor, spin and statistics,
The separation between the mesonic trajectories is nearly the

same as the one between baryonic trajectories suggesting that the same mechanism which breaks the $SU(6)$ symmetry must also be responsible for breaking its supersymmetric extension. Finally, the near linearity of the trajectories implies that the quark binding potential is almost linear, in contrast to quarkonia where a nonrelativistic Schrödinger theory suffices, relativistic quantum mechanics should be applied to light quarks.

In the quark model of Gell-Mann and Zweig, mesons and baryons are $(q\bar{q})$ and (qqq) bound states. So it is natural to conceive of any symmetry between mesons and baryons at the quark level as arising from an effective supersymmetry between an antiquark $\bar{q}$ and bound $(D = qq)$ states or diquarks.

We recall that the quark q, with $s = \frac{1}{2}$ and unitary spin of a flavor $SU(3)$ triplet (3) belongs to the $SU(6)$ sextet (6) representation. The low-lying baryons are then in its (56) representation. Since the (56) is contained in $6 \times 21 =$

$56+70$, the diquark with $s=0$ or 1 must be in the (21) of $SU(6)$. Consequently, the sought for hadronic supersymmetry must transform the $SU(6)$ multiplets (6) and (21), both color antitriplets, into each other. It must therefore be 27-dimensional with 6 fermionic and 21 bosonic states, fitting into $SU(6/21)$. Realization of the $\bar{q}-D$ symmetry, and hence also $q-\bar{D}$ symmetry, will generally transform the meson $q\bar{q}$ not to just baryons qqq and antibaryons $\bar{q}\bar{q}\bar{q}$ but also to exotic mesons $D-\bar{D}$, belonging to the $SU(6)$ representations $1, 35$ and 405. The (1) and (35) are 0^+ and 1^+ mesons while the (405) also includes mesons with spin 2^+ and isospin 2. All the low energy hadrons now sit in the adjoint representation of $SU(6/21)$ with both spin and isospin having values $0, \frac{1}{2}, 1, \frac{3}{2}$ and 2.
The earlier introduction of supersymmetry was in connection with dual string models of hadrons which naturally gave rise to parallel linear meson and baryon trajectories. Unfortunately, these specific Ramond, and Neveu-Schwarz models,

beside being unrealistic, are only relativistic (i.e. Lorentz invariant) in 10 spacetime dimensions. They did not make a comeback till recent years, albeit as Theories of Everything, and their deep connection to division and Jordan algebras which we will not discuss here. Through years the difficult search for a consistent 4-dimensional hadronic superstring theory has continued without significant success, along with attempts such as the large N-limit of QCD.
Under the color group $SU(3)^c$, meson $q\bar{q}$ and diquark ($D = qq$) states transform as

$$qq : \quad 3 \times 3 = \bar{3} + 6 \quad ; \qquad q\bar{q} : \quad 3 \times \bar{3} = 1 + 8 \tag{1}$$

and under the spin flavor $SU(6)$ they transform as

$$qq : \quad 6 \times 6 = 15 + 21 \quad ; \qquad q\bar{q} : \quad 6 \times \bar{6} = 1 + 35 \tag{2}$$

Dimensions of internal degrees of quarks and diquarks are shown in the following table:

	$SU_f(3)$	$SU_s(2)$	$dim.$
q	$\square$	$s=1/2$	$3\times 2=6$
D	$\begin{array}{ll}\square & \square\\ \square & \\ \square & \end{array}$	$s=1$ $s=0$	$6\times 3=18$ $3\times 1=3$

If one writes qqq as qD, then the quantum numbers of D are $\bar{3}$ for color since when combined with q must give a color singlet, and 21 for spin-flavor since combined with color must give antisymmetric wavefunctions. The quantum numbers for $\bar{q}$ are for color, $\bar{3}$, and for spin-flavor, $\bar{6}$. Thus $\bar{q}$ and D have the same quantum numbers (color forces can not distinguish between $\bar{q}$ and D). Therefore there is a dynamic supersymmetry in hadrons with supersymmetric partners

$$\psi = \begin{pmatrix} \bar{q} \\ D \end{pmatrix}, \qquad \bar{\psi} = (q \quad \bar{D}) \tag{3}$$

We can obtain all hadrons by combining ψ and $\bar{\psi}$: mesons are $q\bar{q}$, baryons are qD, and exotics are $D\bar{D}$ states.

A quick review: Inside rotationally excited baryons, QCD leads to the formation of diquarks well separated from the remaining quark. At this separation the scalar, spin-independent, confining part of the effective QCD potential is dominant. Since QCD forces are also flavor-independent, the force between the quark q and the diquark D inside an excited baryon is essentially the same as the one between q and the antiquark $\bar{q}$ inside an excited meson. Thus the approximate spin-flavor independence of hadronic physics expressed by $SU(6)$ symmetry is extended to $SU(6/21)$ supersymmetry, through a symmetry between $\bar{q}$ and D, resulting in the parallelism of mesonic and baryonic Regge trajectories.

Algebraic justification: Color Algebra

We now give an algebraic justification for our remarks above. We will find answers through an algebra built in terms of octonions and their split basis.

The exact, unbroken color group $SU(3)^c$ is the backbone of the strong interaction. It is worthwhile to understand its role in the diquark picture more clearly.

In what follows, we first give a simple description of octonion algebra (also known as Cayley algebra). Later we'll show how to build split octonion algebra that will close into a fermionic Heisenberg algebra. Split octonion algebra will then be shown to produce algebra of color forces in QCD in application to hadronic supersymmetry when the split units and their conjugates become associated with quark and antiquark fields, respectively.

An octonion x is a set of eight real numbers

$$x = (x_0, x_1, \ldots, x_7) = x_0 e_0 + x_1 e_1 + \ldots + x_7 e_7 \qquad (4)$$

that are added like vectors and multiplied according to the rules

$$e_0 = 1, \qquad e_0 e_i = e_i e_0 = e_i, \qquad i = 0, 1, \ldots, 7 \qquad (5)$$

$$e_\alpha e_\beta = -\delta_{\alpha\beta} + \epsilon_{\alpha\beta\gamma} e_\gamma. \quad \alpha, \beta, \gamma = 1, 2, \ldots, 7 \qquad (6)$$

where e_0 is the multiplicative unit element and e_α's are the imaginary octonion units. The structure constants $\epsilon_{\alpha\beta\gamma}$ are completely antisymmetric and take the value 1 for combinations

$$\epsilon_{\alpha\beta\gamma} = (165), (257), (312), (471), (543), (624), (736) \qquad (7)$$

Note that the summation convention is used for repeated indices.
The octonion algebra $\mathcal{C}$ is an algebra defined over the field Q of rational numbers, which as a vector space over Q has dimension 8.

We now give reasons for the incorporation of the octonion algebra for hadronic physics, showing only they, through their split octonionic parts provide the correct description of the color algebra in hadrons. In an earlier publication, Cestmir Burdik and I had shown new diagrammatical multiplication rules for doublets and triplets of octonionic units in our attempt to extend the E_8 group into E_{9+1}, or E_{10} with a hope of building $E_{10} \times E_8 \times E_8$ extended heterotic string theory for incorporation of supergravity using Jordan algebras. First of all, two of the colored quarks in the baryon combine into an anti-triplet $3 \times 3 = \bar{3} + (6)$, and in a nucleon $3 \times \bar{3} = 1 + (8)$. The (6) partner of the diquark and the (8) partner of the nucleon do not exist. In hadron dynamics the only color combinations to consider are $3 \times 3 \rightarrow \bar{3}$ and $\bar{3} \times 3 \rightarrow 1$. These relations imply the existence of split octonion units u_i defined below through a representation of the Grassmann algebra $\{u_i, u_j\} = 0$, $i = 1, 2, 3$. What is a bit strange is that operators u_i, unlike ordinary fermionic operators, are not associative. We also have $\frac{1}{2}[u_i, u_j] = \epsilon_{ijk}\, u_k^*$. The Jacobi identity does not hold since $[u_i, [u_j, u_k]] = -ie_7 \neq 0$, where e_7, anticommute with u_i and u_i^*.

The behavior of various states under the color group is best seen if we use split octonion units defined by

$$u_0 = \frac{1}{2}(1 + ie_7), \qquad u_0^* = \frac{1}{2}(1 - ie_7) \tag{8}$$

$$u_j = \frac{1}{2}(e_j + ie_{j+3}), \qquad u_j^* = \frac{1}{2}(e_j - ie_{j+3}), \quad j = 1, 2, 3 \tag{9}$$

The automorphism group of the octonion algebra is the 14-parameter exceptional group G_2. The imaginary octonion units $e_\alpha (\alpha = 1, ..., 7)$ fall into its 7-dimensional representation.

Under the $SU(3)^c$ subgroup of G_2 that leaves e_7 invariant, u_0 and u_0^* are singlets, while u_j and u_j^* correspond, respectively, to the representations 3 and $\bar{3}$. The multiplication table can now be written in a manifestly $SU(3)^c$ invariant manner (together with the complex conjugate equations):

$$u_0^2 = u_0, \qquad u_0 u_0^* = 0 \tag{10}$$

$$u_0 u_j = u_j u_0^* = u_j, \qquad u_0^* u_j = u_j u_0 = 0 \tag{11}$$

$$u_i u_j = -u_j u_i = \epsilon_{ijk} u_k^* \tag{12}$$

$$u_i u_j^* = -\delta_{ij} u_0 \tag{13}$$

where ϵ_{ijk} is completely antisymmetric with $\epsilon_{ijk} = 1$ for $ijk =$ 123, 246, 435, 651, 572, 714, 367; and zero otherwise. Here, one sees the virtue of octonion multiplication. If we consider the direct products

$$C : \qquad 3 \otimes \bar{3} = 1 + 8 \tag{14}$$

$$G : \qquad 3 \otimes 3 = \bar{3} + 6 \tag{15}$$

for $SU(3)^c$, then these equations show that octonion multiplication gets rid of 8 in $3 \otimes \bar{3}$, while it gets rid of 6 in $3 \otimes 3$. Combining Eq.(**??**) and Eq.(**??**) we find

$$(u_i u_j) u_k = -\epsilon_{ijk} u_0^* \tag{16}$$

Thus the octonion product leaves only the color part in $3 \otimes \bar{3}$ and $3 \otimes 3 \otimes 3$, so that it is a natural algebra for colored quarks. For convenience, we now produce the following multiplication table for the split octonion units:

	u_0	u_0^*	u_k	u_k^*
u_0	u_0	0	u_k	0
u_0^*	0	u_0^*	0	u_k^*
u_j	0	u_j	$\epsilon_{jki} u_i^*$	$-\delta_{jk} u_0$
u_j^*	u_j^*	0	$-\delta_{jk} u_0^*$	$\epsilon_{jki} u_i$

It is worth noting that u_i and u_j^* behave like fermionic annihilation and creation operators:

$$\{u_i, u_j\} = \{u_i^*, u_j^*\} = 0, \quad \{u_i, u_k^*\} = -\delta_{ik} u_0 \tag{17}$$

The quarks, being in the triplet representation of the color group $SU(3)^c$, are represented by the local fields $q_\alpha^i(x)$, where

$i = 1, 2, 3$ is the color index and α the combined spin-flavor index. Antiquarks at point y are color antitriplets $\bar{q}^i_\beta(y)$. Consider the two-body systems

$$C^{\beta i}_{\alpha j} = q^i_\alpha(x_1)\bar{q}^j_\beta(x_2) \tag{18}$$

$$G^{ij}_{\alpha\beta} = q^i_\alpha(x_1)q^j_\beta(x_2) \tag{19}$$

so that C is either a color singlet or color octet, while G is a color antitriplet or a color sextet. Now C contains meson states that are color singlets and hence observable. The octet $q - \bar{q}$ state is confined and not observed as a scattering state. In the case of two-body G states, the antitriplets are diquarks which, inside a hadron can be combined with another triplet quark to give observable, color singlet, three-quark baryon states. The color sextet part of G can only combine with a third quark to give unobservable color octet and color decuplet three-quark states. Hence the hadron dynamics is such that the 8 part of C and the 6 part of G are suppressed. This can best be achieved by the use of the above octonion algebra. The dynamical suppression of the octet and sextet states in Eq.(**??**) and Eq.(**??**) is, therefore, automatically achieved. The split octonion units can be contracted with color indices of triplet or antitriplet fields. For quarks and antiquarks we can define the "transverse" octonions (calling u_0 and u_0^* longitudinal units)

$$q_\alpha = u_i q^i_\alpha = \mathrm{u} \cdot \mathrm{q}_\alpha, \qquad \bar{q}_\beta = u^\dagger_i \bar{q}^j_\beta = -\mathrm{u}^* \cdot \bar{\mathrm{q}}_\beta \tag{20}$$

We find

$$q_\alpha(1)\bar{q}_\beta(2) = u_0 \mathrm{q}_\alpha(1) \cdot \bar{\mathrm{q}}_\beta(2) \tag{21}$$

$$\bar{q}_\alpha(1) q_\beta(2) = u_0^* \bar{\mathrm{q}}_\alpha(1) \cdot \mathrm{q}_\beta(2) \tag{22}$$

$$G_{\alpha\beta}(12) = q_\alpha(1) q_\beta(2) = \mathrm{u}^* \cdot \mathrm{q}_\alpha(1) \times \mathrm{q}_\beta(2) \tag{23}$$

$$G_{\beta\alpha}(21) = q_\beta(2) q_\alpha(1) = \mathrm{u}^* \cdot \mathrm{q}_\beta(2) \times \mathrm{q}_\alpha(1) \tag{24}$$

Because of the anticommutativity of the quark fields, we have

$$G_{\alpha\beta}(12) = G_{\beta\alpha}(21) = \frac{1}{2}\{q_\alpha(1), q_\beta(2)\} \tag{25}$$

If the diquark forms a bound state represented by a field $D_{\alpha\beta}(x)$ at the center-of-mass location x

$$x = \frac{1}{2}(x_1 + x_2) \tag{26}$$

when x_2 tends to x_1 we can replace the argument by x, and we obtain

$$D_{\alpha\beta}(x) = D_{\beta\alpha}(x) \tag{27}$$

so that the local diquark field must be in a symmetric representation of the spin-flavor group. If the latter is taken to be $SU(6)$, then $D_{\alpha\beta}(x)$ is in the 21-dimensional symmetric representation, given by

$$(6 \otimes 6)_s = 21 \tag{28}$$

If we denote the antisymmetric 15 representation by $\Delta_{\alpha\beta}$, we see that the octonionic fields single out the 21 diquark representation at the expense of $\Delta_{\alpha\beta}$. We note that without this color algebra supersymmetry would give antisymmetric configurations as noted by Salam and Strathdee in their possible supersymmetric generalization of hadronic supersymmetry. Using the nonsingular part of the operator product expansion we can write

$$\tilde{G}_{\alpha\beta}(\mathrm{x}_1, \mathrm{x}_2) = D_{\alpha\beta}(x) + \mathrm{r} \cdot \Delta_{\alpha\beta}(x) \tag{29}$$

The fields $\Delta_{\alpha\beta}$ have opposite parity to $D_{\alpha\beta}$; r is the relative coordinate at time t if we take $t = t_1 = t_2$. They play no role in the excited baryon which becomes a bilocal system with the 21- dimensional diquark as one of its constituents. Now consider a three-quark system at time t. The c.m. and relative coordinates are

$$\mathrm{R} = \frac{1}{\sqrt{3}}(\mathrm{r}_1 + \mathrm{r}_2 + \mathrm{r}_3) \tag{30}$$

$$\vec{\rho} = \frac{1}{\sqrt{6}}(2\mathrm{r}_3 - \mathrm{r}_1 - \mathrm{r}_2) \tag{31}$$

$$\mathrm{r} = \frac{1}{\sqrt{2}}(\mathrm{r}_1 - \mathrm{r}_2) \tag{32}$$

giving

$$\mathrm{r}_1 = \frac{1}{\sqrt{3}}\mathrm{R} - \frac{1}{\sqrt{6}}\vec{\rho} + \frac{1}{\sqrt{2}}\mathrm{r} \tag{33}$$

$$\mathrm{r}_2 = \frac{1}{\sqrt{3}}\mathrm{R} - \frac{1}{\sqrt{6}}\vec{\rho} - \frac{1}{\sqrt{2}}\mathrm{r} \tag{34}$$

$$\mathrm{r}_3 = \frac{1}{\sqrt{3}}\mathrm{R} + \frac{2}{\sqrt{6}}\vec{\rho} \tag{35}$$

The baryon state must be a color singlet, symmetric in the three pairs (α, x_1), (β, x_2), (γ, x_3). We find

$$(q_\alpha(1)q_\beta(2))q_\gamma(3) = -u_0^* F_{\alpha\beta\gamma}(123) \tag{36}$$

$$q_\gamma(3)(q_\alpha(1)q_\beta(2)) = -u_0 F_{\alpha\beta\gamma}(123) \tag{37}$$

so that

$$-\frac{1}{2}\{\{q_\alpha(1), q_\beta(2)\}, q_\gamma(3)\} = F_{\alpha\beta\gamma}(123) \tag{38}$$

The operator $F_{\alpha\beta\gamma}(123)$ is a color singlet and is symmetrical in the three pairs of coordinates. We have

$$F_{\alpha\beta\gamma}(123) = B_{\alpha\beta\gamma}(\mathrm{R}) + \vec{\rho} \cdot \mathrm{B}'(\mathrm{R}) + \mathrm{r} \cdot \mathrm{B}''(\mathrm{R}) + C \qquad (39)$$

where C is of order two and higher in $\vec{\rho}$ and r. Because R is symmetric in r_1, r_2 and r_3, the operator $B_{\alpha\beta\gamma}$ that creates a baryon at R is totally symmetrical in its flavor-spin indices. In the $SU(6)$ scheme it belongs to the (56) representation. In the bilocal $q - D$ approximation we have $\mathrm{r} = 0$ so that $F_{\alpha\beta\gamma}$ is a function only of R and $\vec{\rho}$ which are both symmetrical in r_1 and r_2. As before, B' belongs to the orbitally excited 70^- representation of $SU(6)$. The totally antisymmetrical (20) is absent in the bilocal approximation. It would only appear in the trilocal treatment that would involve the 15-dimensional diquarks. Hence, if we use local fields, any product of two octonionic quark fields gives a (21) diquark

$$q_\alpha(\mathrm{R}) q_\beta(\mathrm{R}) = D_{\alpha\beta}(\mathrm{R}) \qquad (40)$$

and any nonassociative combination of three quarks, or a diquark and a quark at the same point give a baryon in the 56^+ representation:

$$(q_\alpha(\mathrm{R})q_\beta(\mathrm{R}))q_\gamma(\mathrm{R}) = -u_0^* B_{\alpha\beta\gamma}(\mathrm{R}) \tag{41}$$

$$q_\alpha(\mathrm{R})(q_\beta(\mathrm{R})q_\gamma(\mathrm{R})) = -u_0 B_{\alpha\beta\gamma}(\mathrm{R}) \tag{42}$$

$$q_\gamma(\mathrm{R})(q_\alpha(\mathrm{R})q_\beta(\mathrm{R})) = -u_0 B_{\alpha\beta\gamma}(\mathrm{R}) \tag{43}$$

$$(q_\gamma(\mathrm{R})q_\alpha(\mathrm{R}))q_\beta(\mathrm{R}) = -u_0^* B_{\alpha\beta\gamma}(\mathrm{R}) \tag{44}$$

The bilocal approximation gives the $(35+1)$ mesons and the 70^- baryons with $\ell = 1$ orbital excitation. If we consider a (28×28) octonionic matrix belonging to $SU(6/22)$:

$$Z = \begin{pmatrix} u_0 M & u_o B & \mathrm{u} \cdot \mathrm{Q} \\ u_0 B^\dagger & u_0 N & \mathrm{u} \cdot \mathrm{D}^* \\ \epsilon \mathrm{u}^* \cdot \mathrm{Q}^\dagger & \epsilon \mathrm{u}^* \cdot \mathrm{D}^\dagger & u_0^* L \end{pmatrix} \tag{45}$$

$$Z = \begin{pmatrix} m \times m & m \times n & m \times 1 \\ n \times m & n \times n & n \times 1 \\ 1 \times m & 1 \times n & 1 \times 1 \end{pmatrix} \tag{46}$$

here ϵ can be 1, -1 or 0. M and N are respectively 6×6 and 21×21 hermitian matrices, B a regular 6×21 matrix, $\mathrm{u} \cdot \mathrm{Q}$ a 6×1 column matrix, $\mathrm{u} \cdot \mathrm{D}^*$ a 21×1 column matrix, and L a 1×1 scalar. Such matrices close under anticommutator operations for $\epsilon = 1$. Matrices Z in general are nonassociative, but for $\epsilon = 0$, when the algebra is no longer semi-simple, the Jacobi identity is satisfied and we obtain a hadronic superalgebra which is an extension of the algebra $SU(6/21)$. Its automorphism group includes $SU(6) \times SU(21) \times SU(3)^c$. Thus color is automatically incorporated.
The automorphism group of this $SU(6/22)$ algebra includes $SU(m) \times SU(n) \times SU(3)^c$. In general for $m/2$ flavors and $n = \frac{1}{2}m(m+1)$ (after some algebra) we have the above Z. If $m = 6$, it includes $SU(6) \times O(3)$. If Q is Majorana and D real, then the group becomes $Osp(m/n) \times SU(3)^c$ with subgroup $Sp(2n, R) \times O(m) \times SU(3)^c$.

A look beyond $SU(3) \times SU(3)$: New Mass Formulae

Based on the flavor $SU(3)$ and its breaking into its $SU(2) \times U(1)$ maximal subgroup of isospin and hypercharge, in 1962 the Gell-Mann-Okubo mass formula illuminated the low lying hadronic spectrum. It led to the pseudoscalar mass formula. The mass formula for the vector mesons presented a more delicate problem since the isospin singlet members of the nine vector mesons, namely the physical ω and ϕ were mixtures of octet and singlet states, involving a mixing angle θ_V as a new parameter. A year later Okubo proposed a model for the determination of this mixing angle by requiring the nine vector mesons to fit into a 3×3 matrix. The group theoretic interpretation of ideal mixing followed soon after with the enlargement of $SU(3)$ to $SU(3)_q \times SU(3)_{\bar{q}}$ (by Gürsey and T.D. Lee), one $SU(3)$ being associated with the quarks and the other $SU(3)$ with the antiquarks that are constituents of the vector mesons. In our approach, the nonet corresponds to the representation $(3, 3)$ of this group. Since the u and

d quarks are much lighter than the strange quark s, the $SU(2)\times SU(2)$ subgroup is not badly broken, so that we must decompose with respect to the subgroup $SU(2)_q \times U(1)_q \times SU(2)_{\bar{q}} \times U(1)_{\bar{q}}$ by using the (I, Y) labels for each $SU(2) \times U(1)$. They are shown in the table 1 below.
With respect to the diagonal $SU(3)$ subgroup the hypercharge Y is the sum of Y_q and $Y_{\bar{q}}$ while the isospin I is zero for ω and ϕ, one for ρ, and $\frac{1}{2}$ for K^* and $\bar{K}^*$. Now, the octet breaking hypothesis involves the octet-singlet mixture given by

$$K = I(I+1) - \frac{Y^2}{4}. \tag{47}$$

For the nonet the energy breaking requires the combination

$$E = E_0 + a(K_q + K_{\bar{q}}) + bK. \tag{48}$$

We find the following assignments shown in the table below:

Particle	I_q	$I_{\bar{q}}$	I	Y_q	$Y_{\bar{q}}$	Y	K_q	$K_{\bar{q}}$	$K_q + K_{\bar{q}}$	K
ω	1/2	1/2	0	1/3	-1/3	0	13/18	13/18	13/9	0
ρ	1/2	1/2	1	1/3	-1/3	0	13/18	13/18	13/9	2
K^*	1/2	0	1/2	1/3	2/3	1	13/18	-1/9	11/18	1/2
$\bar{K}^*$	0	1/2	1/2	-2/3	-1/3	-1	-1/9	13/18	11/18	1/2
ϕ	0	0	0	-2/3	2/3	0	-1/9	-1/9	-2/9	0

Table 1. Particle assignments

Note that the sum $K_q + K_{\bar{q}}$ of the two octet breakings gives equal spacing for the energy levels and degeneracy for ω and ρ. It was shown in Okubo's paper that the rest energy breaking formula leads to a quadratic mass formula when the energy differences are large with respect to the mean energy as in the case of the pseudoscalar mesons and to a linear mass formula when the ratio of the energy splittings to the mean energy is small as in the case of baryons. The vector mesons being nearer in mass to the baryons than the pseudoscalar mesons we can use the linear mass formula as also suggested by the value of the mixing angle being nearer perfect mixing in this case. Then we get

$$m_\omega = \mu + \frac{13}{9}a, \tag{49}$$

$$m_\rho = \mu + \frac{13}{9}a + 2b, \tag{50}$$

$$m_{K^*} = m_{\bar{K}^*} = \mu + \frac{11}{18}a + \frac{1}{2}b, \tag{51}$$

$$m_\phi = \mu - \frac{2}{9}a, \tag{52}$$

leading to the mass sum rule

$$2m_{K^*} = m_\phi + \frac{1}{2}(m_\omega + m_\rho) \tag{53}$$

With the choice

$$\mu = 988.4\ MeV, \qquad a = -142.2\ MeV, \qquad b = -6.5\ MeV \tag{54}$$

we find the following masses as compared to experimental values given in Particle Data Table:

Particle	our result	experiment
m_ω	783	782.65
m_ρ	770	775.49
$m_{K^*} = m_{\bar{K}^*}$	898.25	891.66
m_ϕ	1020	1019.46

Table 2. Calculated masses vs. experiment

and the mass formula Eq.(**??**) gives deviations much less than one percent with above choices. The Eq.(**??**) which we have derived here using purely the $SU(3) \times SU(3)$ group theoretical assignments shown in Table1 is usually written as the squared mass formula

$$4m_{K^*}^2 = 2m_\phi^2 + m_\omega^2 + m_\rho^2 \tag{55}$$

for which the deviations are much larger than the linear formula we wrote down.
It is also important to note that the quantum numbers of Table 1 which forbid the decay of the ϕ into pions, or more generally of the $s\bar{s}$ system into systems involving u and d quarks is consistent with the OZI rule. This rule must be violated in QCD through gluonic intermediate states and yet it is surprisingly well verified, reinforcing the symmetry breaking chain that gives Eq.(**??**).
The next step came from an attempt to put pseudoscalar mesons and vector mesons in a single multiplet. This is natural in a quark model since the lightest quarks u d s have each spin $\frac{1}{2}$, giving 6 states as a representation of the group $SU(6)$. Then the group $SU(6)_q \times SU(6)_{\bar{q}}$ has a $SU(6)$ diagonal subgroup that generalizes the $SU(3)$ of Gell-Mann and Ne'eman through the incorporation of the quark spin associated with $SU(2)$. However, spin is conserved independently of total relativistic angular momentum only for free particles,

so that spin can only combined with flavor quantum numbers like isospin if quarks inside the baryon or meson behave like approximately free fermions. This is a posteriori justified by the asymptotic freedom of the QCD theory. The low lying mesons now fit in the singlet and 35-dimensional adjoint representation of $SU(6)$ with $\eta^{'}$ being the singlet and the eight pseudoscalars and the three $s = 1$ states of the vector nonet completing the 35. Assuming again symmetry breaking to arise from a mixture of $SU(3)$ singlet and the eight member of the octet in the symmetric 35×35 product which contains the 1, 35, 189 and 405 representations of $SU(6)$, one finds the new general formula

$$E = E_0 + \alpha[I(I+1) - \frac{Y^2}{4}] + \beta Y + \gamma s(s+1). \qquad (56)$$

For baryons, E is the mass, for mesons β vanishes and one can approximate E by μ^2 since pseudoscalar mesons are involved. This gives breaking of the (56) baryons ($s = \frac{1}{2}$ octet together with $s = \frac{3}{2}$ decuplet) as well as the (35) mesons. For instance one obtains the mass formula

$$m_{K^*}{}^2 - m_\rho{}^2 = m_K{}^2 - m_\pi{}^2 \tag{57}$$

relating vector and pseudoscalar meson mass splittings and hence showing approximate spin independence of the binding forces in addition to flavor independence implied by $SU(3)$ symmetry.

At this point the baryons which are three quark(qqq) fermionic systems are treated seperately from the ($q\bar{q}$) bosonic mesons. But if the quark binding forces are approximately spin independent they should also be blind to the distinction between fermions with half odd-integer spin and bosons with integer spin. It is this idea that led Miyazawa in the years 1966-68 to propose a fermion-boson symmetry in the hadronic spectrum which later became known as supersymmetry. What are the indications for a broken symmetry for hadrons? First of all low lying mesonic spectrum for $s = 0, 1$ extends from

0.14 GeV to 1.02 GeV while the $s = \frac{1}{2}, \frac{3}{2}$ baryons cover the range 0.94 GeV to 1.67 GeV, so that the two spectra overlap. The mass splittings between adjacent isotopic multiplets are also similar, being of the order of 0.1 GeV in both cases. Hence all low lying hadrons seem to be members of a single supermultiplet with the spin taking the values $0, \frac{1}{2}, 1, \frac{3}{2}$. Another experimental evidence is provided by excited hadronic states with $s \geq 2$. If such states are represented by points on a Chew-Frautschi plot of m^2 versus s, then they fall on parallel linear Regge trajectories. The astounding fact is that the slopes of the meson and baryon trajectories are nearly equal. This is a manifestation of deep and unexpected supersymmetry between excited $(q\bar{q})$ and (qqq) states.
The mathematical expression of supersymmetry arises through a generalization of Lie algebras to superalgebras. When a Lie algebra is $su(n)$ it can be extended to a graded algebra (superalgebra) $su(m/n)$ with even and odd generators, the even generators being paired with commuting (bosonic) parame-

ters and the odd generator with the Grassmann (fermionic) parameters. The algebra can then be exponentiated to the supergroup $SU(m/n)$. This was done by Miyazawa who derived the correct commutation and anticommutation relations for such a superalgebra as well as the generalized Jacobi identity. This discovery antidates the supersymmetry in dual resonance models (Ramond; Neveu and Schwarz 1971) or supersymmetry in quantum field theories (Golfand and Likthman (1971); Volkov and Akulov (1973)) invariant under the super-Poincare' group (Wess and Zumino (1974)) that generalizes special relativity. Miyazawa looked for a supergroup that would contain $SU(6)$ and settled on broken $SU(6/21)$. He showed that an $SU(3)$ singlet-octet of this supergroup leads to a new kind of mass formula relating fermionic and bosonic mass splittings. An example for non strange hadrons is

$$m_{\Delta}{}^{2} - m_{N}{}^{2} = m_{\rho}{}^{2} - m_{\pi}{}^{2} \tag{58}$$

Note that in a potential approximation (in an earlier paper, SC) we proved that

$$m_{\Delta}{}^2 - m_N{}^2 = \frac{9}{8}(m_{\rho}{}^2 - m_{\pi}{}^2) \tag{59}$$

with remarkable accuracy of 1% with experiment.
The emergence of the group $SU(6/21)$ can be understood on the basis of the quark model. The quarks (u, d, s) are associated with the six dimensional representation of $SU(6)$. Assuming that two quarks can form a bound state (this being justified by QCD), the diquarks (qq) belong to the representation 15 or 21. Actually, the $SU(3)$ color coupled with the Pauli principle and the $SU(3)$ singlet nature of (qqq) baryon states gives 21 for the diquark bosonic multiplet. The diquarks and quarks can then combine to give baryonic states which are in the (56) representation of $SU(6)$. It is now clear that the 6 fermionic q-states and the 21 bosonic $\bar{q}\bar{q}$ states together form the 27-dimensional fundamental representation

of the supergroup $SU(6/21)$. The reason we must take the antidiquark to be in the same multiplet as the quark is given by QCD based on color $SU(3)^c$. Each of the 6 colored quark states belongs to the triplet representation of $SU(3)^c$. On the other hand, the diquark is in the $\bar{3}$ representation present in 3×3, so that the antidiquark has the color group representation 3. QCD also gives an attractive force between two quarks (half in strength of the $q\bar{q}$ force) due to one gluon exchange, leading to the formation of qq bound state. Besides the 27-dimensional representation ξ of $SU(6/21)$ there is also the complex conjugate $\overline{27}$ representation consisting of antiquarks and diquarks. The adjoint representation arises from the product $27 \times \overline{27}$ and contains the mesons $q\bar{q}$ $(1+35)$, the baryons qqq $(56+70)$, the antibaryons $\bar{q}\bar{q}\bar{q}$ and the exotic mesons $qq\bar{q}\bar{q}$ which can be regarded as diquark-antidiquark $D\bar{D}$ bound states. Thus QCD provides a basis for formation of a supermultiplet that contains baryons and mesons, the starting point of Miyazawa's model.

The other manifestation of hadronic supersymmetry, namely parallel Regge trajectories for all hadrons is more difficult to relate to group theory. It arises naturally from the string theory of Nambu and Goto (1970) which is associated with the infinite parameter Virasoro algebra (1969) rather than a Lie algebra. The parallel Regge trajectories arise from the flavor and spin independence of the $q\bar{q}$ or qq forces. QCD is certainly flavor independent. As to approximate spin independence, there is mounting evidence that the confining potential is a relativistic scalar rather than the fourth component of a vector potential, although this conclusion is challenged by some authors from a discussion of heavy meson spectra. Now assuming the confining potential is a relativistic scalar, we know from lattice QCD that in its static form it is proportional to the distance r between the quarks. Such a potential for the relativistic two body problem has two consequences. Firstly as shown by Eguchi (1975), and also by Johnson and Thorn (1976), for high rotational excitation the three-quark system tends to a quark-diquark two body system. Secondly, the squared mass of the two body system with a linear potential becomes proportional to the angular momentum J of the system. The first property tells us that the excited baryon can be treated as a two body $q - D$ system ($D = qq$), just like the meson which is a $q - \bar{q}$ sytem. Now, both D and $\bar{q}$ are in the $\bar{3}$ color representation, so that the $q - D$ potential is the same as the $q - \bar{q}$ potential provided we neglect the spin dependence of the forces. The short range force, due to a gluon exchange, is obtained from a vector Coulomb-like potential and is spin dependent. But for high excitation the two constituents have a large separation and the spin-independent confining force takes over, resulting in the approximate equivalence of the Hamiltonian for the $q - \bar{q}$ and $q - D$ systems. The slope of the mother Regge trajectory depends only on the parameter of the confining potential, resulting in parallel linear Regge trajectories for baryons and mesons. The Hamiltonian is also approximately invariant under the transformation of $\bar{q}$ into D, which is a supersymmetry transformation belonging to the supergroup $SU(6/21)$ for the

low lying hadrons.
The string approximation to QCD gives therefore a new type of mass formula

$$m^2 = \alpha^{'-1} J + C = \alpha^{'-1}(J - J_0) \tag{60}$$

valid for both baryons and mesons. Here, the Regge slope $\alpha^{'}$ is of the order of $1(GeV)^{-2}$. This expression gives

$$\Delta m^2 = \alpha^{'-1} \Delta J \tag{61}$$

as for $\Delta J = 1$ we obtain

$$\Delta m^2 = \alpha^{'-1} \tag{62}$$

both for baryons and mesons. Now since $\alpha^{'}$ is the same for the π and ρ trajectories. It is also the same for N and Δ trajectories, giving

$$m_\Delta{}^2 - m_N{}^2 = m_\rho{}^2 - m_\pi{}^2. \tag{63}$$

which is the same as Miyazawa's sum rule obtained from supersymmetry. Here the relation is also valid for any two pairs of points on the same trajectories provided ΔJ is unity. In short Miyazawa hadronic supersymmetry for the low lying hadrons is extended through QCD to the rotationally excited hadronic levels.

Breaking of this supersymmetry has two origins. First the q and qq mass differences as well as mass differences among quarks. This results in different values of the constant J_0 in Eq.(**??**), leading to different intercepts for parallel Regge trajectories. The second breaking comes from the contribution to the potential from one gluon exchange. This potential is a 4-vector and is spin dependent. Since the quark and anti-quark have $s = \frac{1}{2}$ and the diquark has $s = 0$ or 1, the spin dependent part of the $q - qq$ potential is different from that of $q - \bar{q}$, causing supersymmetry breaking. Another consequence is the deviation of the Regge trajectories from linearity for low spin, since the potential is no longer proportional to the distance.

Examples of effective Hamiltonians obtained from a two body Schrödinger-Dirac approximation to the quark-QCD system after elimination of the gluon degrees of freedom will not be presented here due to time limitation. They also exhibit an approximate $SU(6)$ symmetry and $SU(6/21)$ supersymmetry with explicit symmetry-breaking terms.

THANK YOU